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RATIO OF DIFFERENCE OF MEANS AND ITS CONVEXITY

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ABSTRACT:
In this paper, we study on ratio of difference of means. Further some new inequalities are established.
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1. Introduction

In this section, the various means which are essential for the study are defined. The four well-known means are presented by Pappus of Alexandria in his books in the fourth century A.D., which is the main contribution of the ancient Greeks [6]. In Pythagorean School, on the basis of proportion the means are defined as;

For a, b be two positive real numbers, then

(1.1) $A(a, b) = \frac{a+b}{2}$, (1.2) $G(a, b) = \sqrt{ab}$

(1.3) $H(a,b) = \frac{2ab}{a+b}$ (1.4) $C(a,b) = \frac{a^2+b^2}{a+b}$

are respectively called Arithmetic mean, Geometric mean, Harmonic mean and Contra Harmonic mean[1,2].

For a, b be two positive real numbers, then Heron mean is denoted by $H_e(a, b)$ and defined as;

(1.5) $$H_e(a,b) = \frac{a+\sqrt{ab}+b}{3}$$

which occurs in an Egyptian manuscript in the year 1850 B.C. In literature means (1.1) - (1.5) are extensively studied by various researchers and established remarkable inequalities involving them. V. Lokesha and et al, and various others researchers have obtained some interesting and valuable results, an impressive amount of work on generalization of Heron mean, using the generalized Vander Monde's determinants[3,4,7,8].

Let $a=t,\ b=1$ in means (1.1) - (1.5), takes the following forms;

$$A(t,1)=\frac{t+1}{2},\ G(t,1)=\sqrt{t},\ \ H(t,1)=\frac{2t}{t+1},\ \ C(t,1)=\frac{t^2+1}{t+1}\ \text{ and } H_e(t,1)=\frac{t+\sqrt{t}+1}{3}$$

Various researchers have studied several functions, obtained some identities among means and established remarkable mean inequalities. In [5], Jamal Rooin and Mehdi Hassni, introduced the functions *f(x)* and *g(x)*, where

$$f(x)=\frac{a^x-b^x}{c^x-d^x} \text{ and } g(x)=\ln\frac{a^x-b^x}{c^x-d^x}, \text{ for } x\in(-\infty,\infty) \text{ and } a>b\geq c>d>0.$$ Further,

they obtained some results on convexity and application to Refinement of Ky-Fan-type inequalities, Inequalities involving ratio of means.

This motivation made us to introduce the ratio of difference of means and study the convexity of these and to establish some inequalities involving them.

2. Convexity and Some Results

In this section, we introduce the ratio of difference of means as follows:

(2.1) $M_{CAGH}(t)=\dfrac{C(t,1)-A(t,1)}{G(t,1)-H(t,1)}$. (2.2) $M_{CAH_eG}(t)=\dfrac{C(t,1)-A(t,1)}{H_e(t,1)-G(t,1)}$.

(2.3) $M_{CH_eGH}(t)=\dfrac{C(t,1)-H_e(t,1)}{G(t,1)-H(t,1)}$. (2.4) $M_{CAH_eH}(t)=\dfrac{C(t,1)-A(t,1)}{H_e(t,1)-H(t,1)}$

(2.5) $M_{AH_eGH}(t)=\dfrac{A(t,1)-H_e(t,1)}{G(t,1)-H(t,1)}$.

To study the convexity of the ratio of difference of means, we recall the following lemma.

Lemma 1. For $a>b\geq c>d>0$, the function $f(x)=\dfrac{a^x-b^x}{c^x-d^x}$, where $x\in(-\infty,\infty)$ is

(i) Convex, if $ad-bc>0$
(ii) Concave, if $ad-bc<0$ and
(iii) Equality holds, if $ad-bc=0$

Using above lemma, we prove the following Theorem.

Theorem 1: The ratio of difference of means M_{AH_eGH} is convex, if $\sqrt{t} \in \left(\frac{1}{3}, 1\right)$ or $\sqrt{t} \in \left(0, \frac{1}{3}\right) \cup \sqrt{t} \in (1, \infty)$.

Proof: Using the Lemma 1, for $x = 1$ $f(1) = \frac{a-b}{c-d}$

Replace a by $A(t,1)$, b by $H_e(t,1)$, c by $G(t,1)$ and d by $H(t,1)$ and denote $f(t) = M_{AH_eGH}(t)$.

That is $$M_{AH_eGH}(t) = \frac{A(t,1) - H_e(t,1)}{G(t,1) - H(t,1)}$$

Then by lemma 1, $M_{AH_eGH}(t)$ is convex if $AH - H_eG > 0$.

Let us consider, $h(t) = AH - H_eG$

$$= \frac{3t - t^{\frac{3}{2}} - t - t^{\frac{1}{2}}}{3} = \frac{2t - t^{\frac{3}{2}} - t^{\frac{1}{2}}}{3}.$$

Then, $$h^1(t) = \frac{1}{6\sqrt{t}}\left[4\sqrt{t} - 3t - 1\right]$$

$$= \frac{\left(\sqrt{t} - 1\right)\left(1 - 3\sqrt{t}\right)}{6\sqrt{t}}$$

The derivative $h^1(t) \geq 0$, $\begin{cases} \text{if} \quad \sqrt{t} > 1 \quad \text{and} \quad \sqrt{t} < \frac{1}{3} \\ \text{if} \quad \sqrt{t} < 1 \quad \text{and} \quad \sqrt{t} > \frac{1}{3} \end{cases}$

That is $h^1(t) \geq 0$, for (i) $\frac{1}{3} < t < 1$ and $0 < \sqrt{t} < \frac{1}{3}$

(ii) $1 < \sqrt{t}$ or $\sqrt{t} \in \left(\frac{1}{3}, 1\right)$ or $\sqrt{t} \in \left(0, \frac{1}{3}\right) \cup (1, \infty)$

Hence the proof of Theorem 1.

The following results are achieved by using the Theorem 1.

Result 1: Let $A=A(a,b)$, $G=G(a,b)$, $H_e=H_e(a,b)$, $H=H(a,b)$ and $C=C(a,b)$ are means, then following inequality of the ratio of difference of various means

(2.6) $\frac{C-A}{G-H} \leq \frac{2}{3}\frac{C-A}{H_e-G}$ (2.7) $\frac{C-A}{G-H} \leq \frac{1}{6}\frac{A-H_e}{G-H}$ (2.8) $\frac{C-A}{H_e-H} < \frac{C-A}{H_e-G}$

(2.9) $\frac{C-A}{H_e-H} < \frac{C-A}{G-H}$ and (2.10) $\frac{C-A}{G-H} < \frac{C-H_e}{G-H}$.

Proof: The proof of inequality (2.6),

Consider $g_{H_eG-GH}(t) = \dfrac{M''_{H_eG}}{M''_{GH}}$,

Here $M_{H_eG} = H_e - G = \dfrac{t+\sqrt{t}+1}{3} - \sqrt{t} = \dfrac{1}{3}\left(t - 2\sqrt{t} + 1\right)$

Then the second order derivative is $M''_{H_eG} = \dfrac{1}{6t^{3/2}}$

Similarly, if $M_{GH} = G - H = \sqrt{t} - \dfrac{2t}{t+1}$, then $M''_{GH} = \dfrac{16t^{3/2} - (t+1)^3}{4t^{3/2}(t+1)^3}$

Therefore , $g_{H_eG-GH}(t) = \dfrac{M''_{H_eG}}{M''_{GH}} = \dfrac{1}{6t^{\frac{3}{2}}} \; \dfrac{4t^{\frac{3}{2}}(t+1)^3}{16t^{\frac{3}{2}} - (t+1)^3}$

$$g_{H_eG-GH}(t) = \frac{2}{3}\left[\frac{(t+1)^3}{16t^{\frac{3}{2}} - (t+1)^3}\right] \qquad (2.11)$$

On differentiating (2.11) with respect to t

gives, $$g'_{H_eG-GH}(t) = \frac{2}{3}\left[\frac{\left(16t^{\frac{3}{2}} - (t+1)^3\right)3(t+1)^2 - (t+1)^3\left(\frac{48}{2}t^{\frac{1}{2}} - 3(t+1)^2\right)}{\left[16t^{\frac{3}{2}} - (t+1)^3\right]^2}\right]$$

$$= 16\left[\frac{(t+1)}{16t^{\frac{3}{2}} - (t+1)^3}\right]^2 \sqrt{t}\,(t-1)$$

$$= \begin{cases} \geq 0 & for \;\; t > 1 \\ \leq 0 & for \;\; t < 1 \end{cases}$$

This implies that $g_{H_eG-GH}(t)$ is increasing for $t > 1$ that is in $(1,\infty)$ and $g_{H_eG-GH}(t)$ is decreasing in (0,1).

From Definitions of means (1.1) – (1.5), it is easy to obtain $H_e - G < G - H$, also

$$\frac{1}{G-H} < \frac{1}{H_e - G} \quad \text{and} \quad \frac{C-A}{G-H} < \frac{C-A}{H_e - G}.$$

Further, $\beta = g_{H_eG-GH}(1) = \sup_{t\in(0,\infty)} g_{H_eG-GH}(t) = g_{H_eG-GH}(t) = \dfrac{2}{3}$

Hence the proof of inequality $\dfrac{C-A}{G-H} \leq \dfrac{2}{3}\dfrac{C-A}{H_e - G}$.

With similar arguments yields the proof of inequalities (2.7) – (2.10).

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