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# ON THE STABILITY OF A FOUR SPECIES: A PREY-PREDATOR-HOST-COMMENSAL-MUTUAL-SYN ECO-SYSTEM-IV (PREY WASHED OUT STATE)

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**Abstract:**
This paper deals with an investigation on a four Species Syn-Ecological System (Prey Washed out State). The System comprises of a prey ($S_1$) , a predator ($S_2$) that survives upon $S_1$, two hosts $S_3$ and $S_4$ for which $S_1$, $S_2$ are commensal respectively i.e., $S_3$ and $S_4$ benefit $S_1$ and $S_2$ respectively, without getting effected either positively or adversely. Further $S_3$ and $S_4$ are mutual. The model equations of the system constitute a set of four first order non-linear ordinary differential coupled equations. In all, there are sixteen equilibrium points. Criteria for the asymptotic stability of one of the sixteen equilibrium points: the prey washed out state is established. The system would be stable if all the characteristic roots are negative, in case they are real, and have negative real parts, in case they are complex. The linearised equations for the perturbations over the equilibrium point are analyzed to establish the criteria for stability and the trajectories illustrated.

**Keywords**: Commensal, Eco-system, Equilibrium points, Host, Mutual, Prey, Predator, Stability.

## 1. INTRODUCTION

Mathematical modeling of ecosystems was initiated in 1925 by Lotka [10] and by Volterra [17]. The general concepts of modeling have been presented in the treatises of Meyer [11], Kushing [7] and Kapur [5, 6]. K. Lakshminarayan and N.Ch. Pattabhi Ramacharyulu [8] studied the two species prey-predator ecological models incorporating a partial cover for the prey and alternate food for the predator. These authors have also analysed a prey-predator model with alternative food for the predator, harvesting of both the species [9]. The study on competitive eco-systems of two and three species with limited and unlimited resources was done by N.C. Srinivas [16]. R. Archana Reddy [1, 2] and B. Bhaskara Rama Sharma [3] investigated on interacting species with harvesting of both the species at constant rate and competitive eco-systems with time delay, employing analytical and numerical techniques. Further study on the stability of a Host – a flourishing commensal species pair with limited resources was done by N. Phani Kumar, N. Seshagiri Rao and N.Ch. Pattabhi Ramacharyulu [12]. The stability analysis of a four species eco-system with the interaction between $S_3$ and $S_4$ is neutralism was considered by B. Hari Prasad and N.Ch. Pattabhi Ramacharyulu [4].

Following this N. Shanker, K. Lakshminarayan and N.Ch. Pattabhi Ramacharyulu studied stability analysis of a four species eco-system with the interaction between $S_3$ and $S_4$ being mutual [13, 14,15].

The present investigation is on an analytical study of a four species ($S_1$, $S_2$, $S_3$, $S_4$) Prey-Predator-Host-Commensal-Mutual-Syn Eco-System. Fig.1 shows a Schematic Sketch of the system under investigation. In all sixteen equilibrium points are identified based on model equations and the stability analysis is carried out only for the prey washed out state.

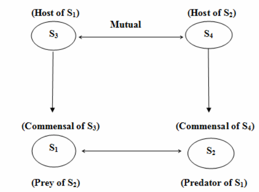


**Fig. 1 Schematic Sketch of the Syn Eco-System under investigation**

## 2. NOTATION ADOPTED

| | | |
|---|---|---|
| $N_1(t)$ | : | The population of the prey species ($S_1$) |
| $N_2(t)$ | : | The population of the predator species ($S_2$) |
| $N_3(t)$ | : | The population of the host species ($S_3$) of the prey ($S_1$) |
| $N_4(t)$ | : | The population of the host ($S_4$) of the predator ($S_2$) |
| t | : | Time instant |
| $a_1, a_2, a_3, a_4$ | : | Natural growth rates of $S_1$, $S_2$, $S_3$, $S_4$ |
| $a_{11}, a_{22}, a_{33}, a_{44}$ | : | Self inhibition coefficients of $S_1$, $S_2$, $S_3$, $S_4$ |
| $a_{12}, a_{21}$ | : | Interaction (prey-predator) coefficients of $S_1$ due to $S_2$ and $S_2$ due to $S_1$ |
| $a_{13}$ | : | Coefficient of commensalism of $S_3$ towards $S_1$ |
| $a_{24}$ | : | Coefficient of commensalism of $S_4$ towards $S_2$ |
| $a_{34}$ | : | Coefficient of mutualism of $S_4$ towards $S_3$ |
| $a_{43}$ | : | Coefficient of mutualism of $S_3$ towards $S_4$ |
| $K_i = \dfrac{a_i}{a_{ii}}$ | : | *Carrying* capacity of $S_i$, i=1,2,3,4 |

Further the variables $N_1$, $N_2$, $N_3$ and $N_4$ are non-negative and the model parameters $a_1$, $a_2$, $a_3$, $a_4$, $a_{11}$, $a_{22}$, $a_{33}$, $a_{44}$, $a_{12}$, $a_{21}$, $a_{13}$, $a_{24}$, $a_{34}$, $a_{43}$ are assumed to be non-negative constants.

## 3. BASIC MODEL EQUATIONS

The model equations for the growth rates of $S_1$, $S_2$, $S_3$, $S_4$ are

$$\frac{dN_1}{dt} = a_1N_1 - a_{11}N_1^2 - a_{12}N_1N_2 + a_{13}N_1N_3 \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \tag{3.1}$$

$$\frac{dN_2}{dt} = a_2N_2 - a_{22}N_2^2 + a_{21}N_1N_2 + a_{24}N_2N_4 \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \tag{3.2}$$

$$\frac{dN_3}{dt} = a_3N_3 - a_{33}N_3^2 + a_{34}N_3N_4 \tag{3.3}$$

$$\frac{dN_4}{dt} = a_4N_4 - a_{44}N_4^2 + a_{43}N_3N_4 \tag{3.4}$$

## 4. EQUILIBRIUM STATES

The system under investigation has sixteen equilibrium states defined by

$$\frac{dN_i}{dt} = 0, \qquad i = 1, 2, 3, 4 \tag{4.1}$$

are given in the following table.

Table I: EQUILIBRIUM STATES

| S.No. | EQUILIBRIUM STATES | EQUILIBRIUM POINT |
|---|---|---|
| 1 | Fully washed out state | $\overline{N_1}=0, \overline{N_2}=0, \overline{N_3}=0, \overline{N_4}=0$ |
| 2 | Only the prey $S_1$ survives | $\overline{N_1}=\frac{a_1}{a_{11}}, \overline{N_2}=0, \overline{N_3}=0, \overline{N_4}=0$ |
| 3 | Only the predator $S_2$ survives | $\overline{N_1}=0, \overline{N_2}=\frac{a_2}{a_{22}}, \overline{N_3}=0, \overline{N_4}=0$ |
| 4 | Only the host ($S_3$) of $S_1$ survives | $\overline{N_1}=0, \overline{N_2}=0, \overline{N_3}=\frac{a_3}{a_{33}}, \overline{N_4}=0$ |
| 5 | Only the host ($S_4$) of $S_2$ survives | $\overline{N_1}=0, \overline{N_2}=0, \overline{N_3}=0, \overline{N_4}=\frac{a_4}{a_{44}}$ |
| 6 | Prey ($S_1$) and the predator ($S_2$) survives | $\overline{N_1}=\frac{a_1a_{22}-a_2a_{12}}{a_{11}a_{22}+a_{12}a_{21}}, \overline{N_2}=\frac{a_2a_{11}+a_1a_{21}}{a_{11}a_{22}+a_{12}a_{21}}, \overline{N_3}=0, \overline{N_4}=0$ |
| 7 | Predator ($S_2$) and the host ($S_4$) of $S_2$ washed out | $\overline{N_1}=\frac{a_1a_{33}+a_3a_{13}}{a_{11}a_{33}}, \overline{N_2}=0, \overline{N_3}=\frac{a_3}{a_{33}}, \overline{N_4}=0$ |
| 8 | Predator ($S_2$) and the host ($S_3$) of $S_1$ washed out | $\overline{N_1}=\frac{a_1}{a_{11}}, \overline{N_2}=0, \overline{N_3}=0, \overline{N_4}=\frac{a_4}{a_{44}}$ |
| 9 | Prey ($S_1$) and the host ($S_4$) of $S_2$ washed out | $\overline{N_1}=0, \overline{N_2}=\frac{a_2}{a_{22}}, \overline{N_3}=\frac{a_3}{a_{33}}, \overline{N_4}=0$ |
| 10 | Prey ($S_1$) and the host($S_3$) of $S_1$ washed out | $\overline{N_1}=0, \overline{N_2}=\frac{a_2a_{44}+a_4a_{24}}{a_{22}a_{44}}, \overline{N_3}=0, \overline{N_4}=\frac{a_4}{a_{44}}$ |
| 11 | Prey ($S_1$) and the predator ($S_2$) washed out | $\overline{N_1}=0, \overline{N_2}=0, \overline{N_3}=\frac{\alpha_2}{\alpha_1}, \overline{N_4}=\frac{\alpha_3}{\alpha_1}$<br>where $\alpha_1=a_{33}a_{44}-a_{34}a_{43}$, $\alpha_2=a_3a_{44}+a_4a_{34}$<br>$\alpha_3=a_4a_{33}+a_3a_{43}$ |
| 12 | Only the host ($S_4$) of $S_2$ washed out | $\overline{N_1}=\frac{\beta_2}{\beta_1}, \overline{N_2}=\frac{\beta_3}{\beta_1}, \overline{N_3}=\frac{a_3}{a_{33}}, \overline{N_4}=0$<br>*where* $\beta_1=a_{33}(a_{11}a_{22}+a_{12}a_{21})$<br>$\beta_2=a_1a_{22}a_{33}+a_3a_{13}a_{22}-a_2a_{12}a_{33}$<br>$\beta_3=a_2a_{11}a_{33}+a_1a_{21}a_{33}+a_3a_{13}a_{21}$ |
| 13 | Only the host($S_3$) of $S_1$ washed out | $\overline{N_1}=\frac{\theta_2}{\theta_1}, \overline{N_2}=\frac{\theta_3}{\theta_1}, \overline{N_3}=0, \overline{N_4}=\frac{a_4}{a_{44}}$<br>*where* $\theta_1=a_{44}(a_{11}a_{22}+a_{12}a_{21})$<br>$\theta_2=a_1a_{22}a_{44}-a_2a_{12}a_{44}-a_4a_{12}a_{24}$<br>$\theta_3=a_2a_{11}a_{44}+a_4a_{11}a_{24}+a_1a_{21}a_{44}$ |
| 14 | Only the Predator ($S_2$)washed out | $\overline{N_1}=\frac{\psi}{a_{11}\alpha_1}, \overline{N_2}=0, \overline{N_3}=\frac{\alpha_2}{\alpha_1}, \overline{N_4}=\frac{\alpha_3}{\alpha_1}$ *where* $\psi=a_1\alpha_1+a_{13}\alpha_3$ |

| | | |
|---|---|---|
| 15* | Only the prey ($S_1$) washed out | $\overline{N_1}=0, \overline{N_2}=\frac{\delta}{a_{22}\alpha_1}, \overline{N_3}=\frac{\alpha_2}{\alpha_1}, \overline{N_4}=\frac{\alpha_3}{\alpha_1}$ where $\delta = a_2\alpha_1 + a_3a_{24}a_{43} + a_4a_{24}a_{33}$ |
| 16 | The co-existent state (or) Normal steady state | $\overline{N_1}=\frac{\sigma_2}{\sigma_1}, \overline{N_2}=\frac{\sigma_3}{\sigma_1}, \overline{N_3}=\frac{\alpha_2}{\alpha_1}, \overline{N_4}=\frac{\alpha_3}{\alpha_1}$ *where* $\sigma_1=(a_{11}a_{22}+a_{12}a_{21})\alpha_1$ $\sigma_2=(-a_1a_{22}+a_2a_{12})\alpha_1+a_3(a_{12}a_{24}a_{43}-a_{13}a_{22}a_{44}) + a_4(a_{12}a_{24}a_{33}-a_{13}a_{22}a_{34})$ $\sigma_3=(a_1a_{21}+a_2a_{11})\alpha_1+a_3(a_{11}a_{24}a_{43}+a_{13}a_{21}a_{44}) + a_4(a_{11}a_{24}a_{33}+a_{13}a_{21}a_{34})$ |

The stability analysis of the fully washed out state (S.No.1), prey and predator washed out states (S.No's. 4, 5, and 11) and also the predator washed out state (S.No.14) has been carried by the present author [13, 14, 15]. The present paper deals with the stability of prey washed out state (marked *) of the above table only. The stability of the other Equilibrium states will be presented in the forthcoming communications.

## 5. STABILITY OF THE PREY WASHED OUT STATE (Sl. No. 15 in the above table)

To discuss the stability of equilibrium point $\overline{N_1}=0, \overline{N_2}\neq 0, \overline{N_3}\neq 0, \overline{N_4}\neq 0$

*Let us consider small deviations* $u_1(t),\ u_2(t),\ u_3(t),\ u_4(t)$ *from the steady state*

i.e., $N_i(t)=\overline{N}_i+u_i(t)\ ,\qquad i=1,2,3,4\sqrt{b^2-4ac}$ (5.1)

Where $u_i(t)$ is a small perturbation in the species $S_i$.

Substituting (5.1) in (3.1), (3.2), (3.3), (3.4) and neglecting products and higher *powers of* $u_1,\ u_2,\ u_3,\ u_4$ we get

$$\frac{du_1}{dt}=\left(a_1-a_{12}\overline{N_2}+a_{13}\overline{N_3}\right)u_1 \tag{5.2}$$

$$\frac{du_2}{dt}=a_{21}\overline{N}_2u_1-a_{22}\overline{N}_2u_2+a_{24}\overline{N}_2u_4 \tag{5.3}$$

$$\frac{du_3}{dt}=-a_{33}\overline{N}_3u_3+a_{34}\overline{N}_3u_4 \tag{5.4}$$

$$\frac{du_4}{dt}=a_{43}\overline{N}_4u_3-a_{44}\overline{N}_4u_4 \tag{5.5}$$

The characteristic equation of which is

$$\left[\lambda-\left(a_1-a_{12}\overline{N_2}+a_{13}\overline{N_3}\right)\right]\left[\lambda+a_{22}\overline{N}_2\right]\left[\lambda^2+\lambda\left(a_{33}\overline{N}_3+a_{44}\overline{N}_4\right)+\left(a_{33}a_{44}-a_{34}a_{43}\right)\overline{N_3}\,\overline{N_4}\right]=0 \tag{5.6}$$

If $a_1+a_{13}\overline{N_3} < a_{12}\overline{N_2}$ and $a_{33}a_{44} > a_{34}a_{43}$ all the four roots are negative, hence the state is *Stable*. Otherwise the State is *Unstable*.

The solutions of the equations (5.1.2), (5.1.3), (5.1.4), (5.1.5) are

$$u_1=u_{10}e^{\beta t} \tag{5.7}$$

$$u_2=Le^{\beta t}+ARe^{-\lambda_1 t}+BSe^{-\lambda_2 t}+[u_{20}-L-AR-BS]e^{-a_{22}\overline{N_2}t} \tag{5.8}$$

$$\mathrm{u}_3=\mathrm{P}\mathrm{e}^{-\lambda_1 \mathrm{t}}+\mathrm{Q}\mathrm{e}^{-\lambda_2 \mathrm{t}} \tag{5.9}$$

$$u_4=Re^{-\lambda_1 t}+Se^{-\lambda_2 t} \tag{5.10}$$

where $P=\dfrac{-u_{30}s_1+u_{30}a_{44}\overline{N_4}+u_{40}a_{34}\overline{N_3}}{\lambda_2-\lambda_1}$, $Q=\dfrac{-u_{30}s_2+u_{30}a_{44}\overline{N_4}+u_{40}a_{34}\overline{N_3}}{\lambda_1-\lambda_2}$,

$R=\dfrac{-u_{40}s_1+u_{40}a_{33}\overline{N_3}+u_{30}a_{43}\overline{N_4}}{\lambda_2-\lambda_1}$, $S=\dfrac{-u_{40}s_2+u_{40}a_{33}\overline{N_3}+u_{40}a_{43}\overline{N_4}}{\lambda_1-\lambda_2}$,

$L=\dfrac{u_{10}a_{21}\overline{N_2}}{a_1+\left(a_{22}-a_{12}\right)\overline{N_2}+a_{13}\overline{N_3}}$, $\beta=\left(a_1-a_{12}\overline{N_2}+a_{13}\overline{N_3}\right)t$,

$A=\dfrac{a_{24}\overline{N_2}}{a_{22}\overline{N_2}-\lambda_1}$ and $B=\dfrac{a_{24}\overline{N_2}}{a_{22}\overline{N_2}-\lambda_2}$

There would arise in all 576 cases depending upon the ordering of the magnitudes of the growth rates $a_1, a_2, a_3, a_4$ and the *initial values of the perturbations* $u_{10}(t), u_{20}(t), u_{30}(t), u_{40}(t)$ *of the* species $S_1$, $S_2$, $S_3$, and $S_4$. Of these 576 situations some typical variations are illustrated in figures 2 to 13 through respective solution curves that would facilitate to make some reasonable observations and the conclusions are presented here.

## 6. CONCLUSIONS OF THE PERTURBATION GRAPHS

**Case A :** when $a_1+a_{13}\overline{N_3} < a_{12}\overline{N_2}$ and $a_{33}a_{44} > a_{34}a_{43}$

In this case all the roots are negative. Three typical variations are considered here under cases A1, A2, A3.

**Case A1:** If $u_{10} < u_{30} < u_{20} < u_{40}$ and $a_3 < a_2 < a_1 < a_4$

In this case the host ($S_3$) of the prey ($S_1$) has the least natural growth rate, and the prey ($S_1$) has the least initial population strength. The prey ($S_1$) initially dominates over the host ($S_3$) of the prey ($S_1$) and over the predator ($S_2$) till the time instant $t^*_{13}$ and $t^*_{12}$ respectively (obtained on solving equations 5.7 & 5.9 and 5.7 & 5.8 respectively) and thereafter the dominance is reversed as shown in Fig.2.

**Case A2:** If $u_{20} < u_{40} < u_{10} < u_{30}$ and $a_4 < a_2 < a_3 < a_1$

In this case the host ($S_4$) of the predator ($S_2$) has the least natural growth rate, and the predator ($S_2$) has the least initial population strength. The predator ($S_2$) initially dominates over the host ($S_4$) of the predator ($S_2$) till the time instant $t^*_{24}$ (obtained on solving equations 5.8 & 5.10) and thereafter the dominance is reversed. Also the prey ($S_1$) initially dominates over the host ($S_3$) of the prey ($S_1$) till the time instant $t^*_{13}$ (obtained on solving equations 5.7 & 5.9) and thereafter the dominance is reversed as shown in Fig.3.

**Case A3:** If $u_{40} < u_{10} < u_{30} < u_{20}$ and $a_2 < a_4 < a_1 < a_3$

In this case the predator ($S_2$) has the least natural growth rate, and the host ($S_4$) of the predator ($S_2$) has the least initial population strength. The host ($S_3$) of the prey ($S_1$) initially dominates over host ($S_4$) of the predator ($S_2$) till the time instant $t^*_{34}=\dfrac{1}{\left(\lambda_1-\lambda_2\right)}\log\left(\dfrac{P-R}{S-Q}\right)$ and thereafter the dominance is reversed. Also the prey ($S_1$) initially dominates over the host ($S_4$) of the predator ($S_2$) till the time instant $t^*_{14}$ (obtained on solving equations 5.7 & 5.10) and thereafter the dominance is reversed. And also the predator ($S_2$) dominates over the host ($S_4$) of the predator ($S_2$) till the time instant $t^*_{24}$ (obtained on solving equations 5.7 & 5.8) and thereafter the dominance is reversed as shown in Fig.4.

**Case B :** when $a_1 + a_{13}\overline{N_3} < a_{12}\overline{N_2}$ and $a_{33}a_{44} < a_{34}a_{43}$

In this case one root is positive and the other roots are negative. Three typical variations are considered here under cases B1, B2, B3.

**Case B1:** If $u_{30} < u_{40} < u_{10} < u_{20}$ and $a_4 < a_3 < a_1 < a_2$

In this case the host ($S_4$) of the predator ($S_2$) has the least natural growth rate, and the host ($S_3$) of the prey ($S_1$) has the least initial population strength. The host ($S_3$) of the prey ($S_1$) initially dominates over the host ($S_4$) of the predator ($S_2$) , the prey ($S_1$) and the predator ($S_2$) till the time instant $t_{34}^* = \frac{1}{(\lambda_1 - \lambda_2)}\log\left(\frac{P-R}{S-Q}\right)$, $t_{31}^*$ and $t_{32}^*$ respectively (obtained on solving equations 5.7 & 5.9 and 5.8 & 5.9 respectively) and thereafter the dominance is reversed as shown in Fig.5.

**Case B2:** If $u_{40} < u_{30} < u_{10} < u_{20}$ and $a_3 < a_1 < a_4 < a_2$

In this case the host ($S_3$) of the prey ($S_1$) has the least natural growth rate, and the host ($S_4$) of the predator ($S_2$) has the least initial population strength. The host ($S_3$) of the prey ($S_1$) initially dominates over the prey ($S_1$) and the predator ($S_2$) till the time instant $t_{31}^*$ , $t_{32}^*$ respectively (obtained on solving equations 5.7 & 5.9, 5.8 & 5.9 respectively) and thereafter the dominance is reversed. Also the host ($S_4$) of the predator ($S_2$) initially dominates over the prey ($S_1$) till the time instant $t_{41}^*$ (obtained on solving equations 5.7 & 5.10), thereafter the dominance is reversed as shown in Fig.6.

**Case B3:** If $u_{10} < u_{20} < u_{30} < u_{40}$ and $a_1 < a_4 < a_2 < a_3$

In this case the prey ($S_1$) has the least natural growth rate and least initial population strength. The host ($S_3$) of the prey ($S_1$) initially dominates over the host ($S_4$) of the predator ($S_2$) till the time instant $t_{34}^* = \frac{1}{(\lambda_1 - \lambda_2)}\log\left(\frac{P-R}{S-Q}\right)$ and thereafter the dominance is reversed. Also the predator ($S_2$) initially dominates over the host ($S_4$) of the predator ($S_2$) till the time instant $t_{24}^*$ (obtained on solving equations 5.8 & 5.10) and thereafter the dominance is reversed as shown in Fig.7.

**Case C :** when $a_1 + a_{13}\overline{N_3} > a_{12}\overline{N_2}$ and $a_{33}a_{44} > a_{34}a_{43}$

In this case one root is positive and the other roots are negative. Three typical variations are considered here under cases C1, C2, C3.

**Case C1:** If $u_{10} < u_{40} < u_{20} < u_{30}$ and $a_4 < a_2 < a_1 < a_3$

In this case the host ($S_4$) of the predator ($S_2$) has the least natural growth rate, and the prey ($S_1$) has the least initial population strength. The prey ($S_1$) initially dominates over the host ($S_4$) of the predator ($S_2$), predator ($S_2$), and the host ($S_3$) of the prey ($S_1$) till the time instant $t_{14}^*$ , $t_{12}^*$ and $t_{13}^*$ respectively (obtained on solving equations 5.7 & 5.10, 5.7 & 5.8 and 5.7 & 5.9 respectively) and thereafter the dominance is reversed as shown in Fig.8.

**Case C2:** If $u_{20} < u_{30} < u_{10} < u_{40}$ and $a_2 < a_4 < a_1 < a_3$

In this case the predator ($S_2$) has the least natural growth rate, and the least initial population strength. The prey ($S_1$) initially dominates over the host ($S_4$) of the predator ($S_2$) till the time instant $t_{14}^{*}$ (obtained on solving equations 5.7 & 5.10) and thereafter the dominance is reversed. Also the host ($S_3$) of the prey ($S_1$) initially dominates over the host ($S_4$) of the predator ($S_2$) till the time instant $t_{34}^{*}=\frac{1}{(\lambda_1-\lambda_2)}\log\left(\frac{P-R}{S-Q}\right)$, thereafter the dominance is reversed as shown in Fig.9.

**Case C3:** If $u_{40}<u_{30}<u_{20}<u_{10}$ and $a_1<a_3<a_4<a_2$
In this case the prey ($S_1$) has the least natural growth rate and the host ($S_4$) of the predator ($S_2$) has the least initial population strength. The host ($S_4$) of the predator ($S_2$) initially dominates over the host ($S_3$) of the prey ($S_1$) till the time instant $t_{43}^{*}=\frac{1}{\lambda_1-\lambda_2}\log\left(\frac{R-P}{Q-S}\right)$ and thereafter the dominance is reversed as shown in Fig.10.

**Case D :** when $a_1+a_{13}\overline{N_3}>a_{12}\overline{N_2}$ and $a_{33}a_{44}<a_{34}a_{43}$
In this case two roots are positive and the other two roots are negative. Three typical variations are considered here under cases D1, D2, D3.

**Case D1:** If $u_{10}<u_{30}<u_{40}<u_{20}$ and $a_3<a_1<a_2<a_4$
In this case the host ($S_3$) of the prey ($S_1$) has the least natural growth rate, and the prey ($S_1$) has the least initial population strength. The host ($S_3$) of the prey ($S_1$) initially dominates over the host ($S_4$) of the predator ($S_2$) and the predator ($S_2$) till the time instant $t_{34}^{*}=\frac{1}{(\lambda_1-\lambda_2)}\log\left(\frac{P-R}{S-Q}\right)$ , $t_{32}^{*}$ (obtained on solving equations 5.8 & 5.9) and thereafter the dominance is reversed. Also the prey ($S_1$) initially dominates over the host ($S_4$) of the predator ($S_2$) and the predator ($S_2$) till the time instant $t_{14}^{*},t_{12}^{*}$ (obtained on solving equations 5.7 & 5.10, 5.7 & 5.8 respectively) and thereafter the dominance is reversed as shown in Fig.11.

**Case D2:** If $u_{20}<u_{10}<u_{40}<u_{30}$ and $a_4<a_1<a_2<a_3$
In this case the host ($S_4$) of the predator ($S_2$) has the least natural growth rate, and the predator ($S_2$) has the least initial population strength. The prey ($S_1$) initially dominates over the host ($S_4$) of the predator ($S_2$) and the host ($S_3$) of the prey ($S_1$) till the time instant $t_{14}^{*}$ and $t_{13}^{*}$ respectively (obtained on solving equations 5.7 & 5.10, 5.7 & 5.9 respectively) and thereafter the dominance is reversed as shown in Fig.12.

**Case D3:** If $u_{30}<u_{40}<u_{20}<u_{10}$ and $a_1<a_2<a_3<a_4$
In this case the prey ($S_1$) has the least natural growth rate and the host ($S_3$) of the prey ($S_1$) has the least initial population strength. The predator ($S_2$) initially dominates over the prey ($S_1$) and the host ($S_3$) of the prey ($S_1$) till the time instant $t_{21}^{*}$ and $t_{23}^{*}$ (obtained on solving equations 5.7 & 5.8, 5.8 & 5.9 respectively) and thereafter the dominance is reversed. Also the host ($S_4$) of the predator ($S_2$) initially dominates over the prey ($S_1$) till the time instant $t_{41}^{*}$ (obtained on solving equations 5.8 & 5.10) and thereafter the dominance is reversed as shown in Fig.13.

## 7. TRAJECTORIES OF PERTURBATIONS

The trajectories in $u_1 - u_2,\ u_1 - u_3,\ u_1 - u_4$ planes are

$$u_1 = L\left(\frac{u_1}{u_{10}}\right) + AR\left(\frac{u_1}{u_{10}}\right)^{\frac{-\lambda_1}{\beta}} + BS\left(\frac{u_1}{u_{10}}\right)^{\frac{-\lambda_2}{\beta}} + \left(u_{20} - L - AR - BS\right)\left(\frac{u_1}{u_{10}}\right)^{\frac{-a_{22}\overline{N_2}}{\beta}} ,$$

$$u_3 = P\left(\frac{u_1}{u_{10}}\right)^{\frac{-\lambda_1}{\beta}} + Q\left(\frac{u_1}{u_{10}}\right)^{\frac{-\lambda_2}{\beta}} \quad \text{and} \quad u_4 = R\left(\frac{u_1}{u_{10}}\right)^{\frac{-\lambda_1}{\beta}} + S\left(\frac{u_1}{u_{10}}\right)^{\frac{-\lambda_2}{\beta}} \text{ respectively.}$$

Similarly the trajectories in $u_2 - u_3$ , $u_2 - u_4$ and $u_3 - u_4$ can be found.

## 8. GRAPHS OF THE PERTURBATION

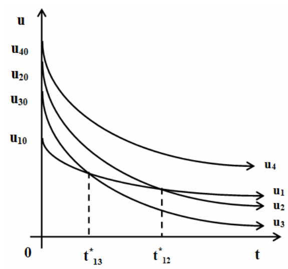

Fig. 2

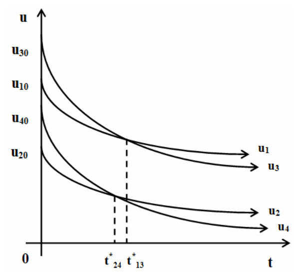

Fig. 3

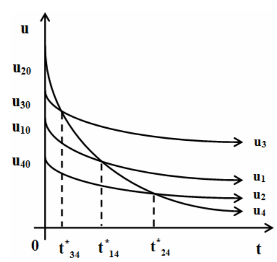

Fig. 4

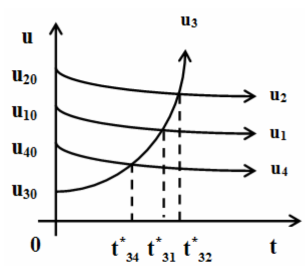

Fig. 5

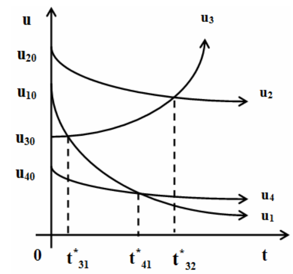

Fig. 6

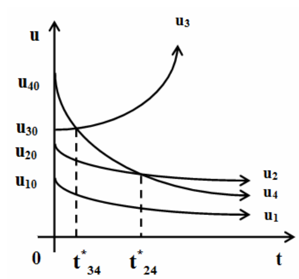

Fig. 7

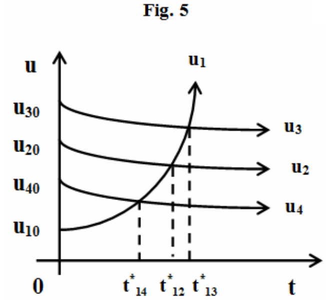

Fig. 8

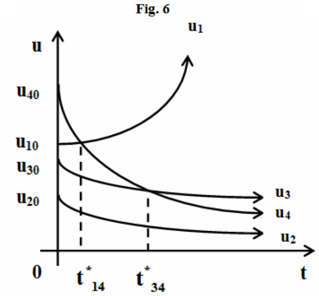

Fig. 9

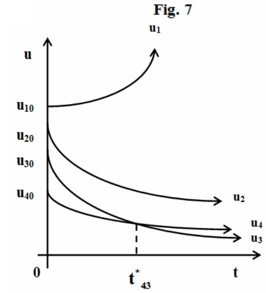

Fig. 10

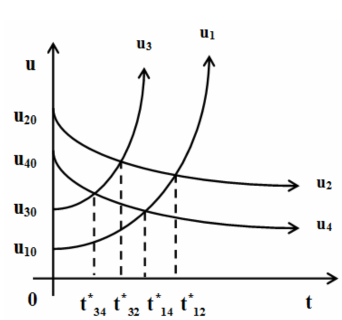

Fig. 11

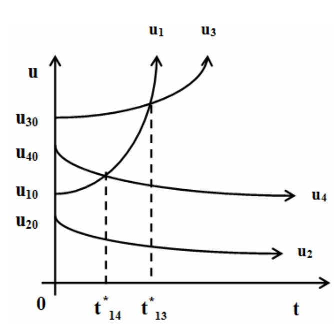

Fig. 12

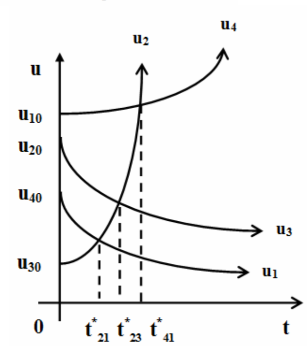

Fig. 13

## References

[1] Archana Reddy R., 2009, “On the stability of some Mathematical Models in Biosciences – Interacting Species,” Ph.D. thesis, J.N.T.U.

[2] Archana Reddy R., Pattabhi Ramacharyulu N. Ch., and Krishna Gandhi B., 2007, “A stability analysis of two competitive interacting species with harvesting of both the species at a constant rate,” *International journal of scientific computing* (1), p. 57-68.

[3] Bhaskara Rama Sharma B., 2009, “Some Mathematical Models in Competitive Eco- Systems,” Ph.D. thesis, Dravidian University.

[4] Hari Prasad B. and Pattabhi Ramacharyulu. N. Ch., “On the stability of a four species: A Prey-Predator-Host-Commensal-Syn Eco-System-VI (Predator Washed out states),” *International eJournal of Mathematics and Engineering*, Volume 1, Issue III, p. 398-410.

[5] Kapur J.N., 1985, “Mathematical Modelling in Biology and Medicine,” affiliated *east west*.

[6] Kapur J.N., 1985, “Mathematical Modelling,” *Wiley Easter*.

[7] Kushing J.M., 1977, “Integro-Differential Equations and Delay Models in Population Dynamics, Lecture Notes in Bio-Mathematics,” *Springer Verlag*, 20.

[8] Lakshmi Narayan K., 2005, “A Mathematical Study of a Prey-Predator Ecological Model with a Partial Cover for the Prey and Alternate Food for the Predator,” Ph.D. thesis, J.N.T.U.

[9] Lakshmi Narayan K., and Pattabhi Ramacharyulu N. Ch., 2008, “A Prey-Predator model with cover for prey and alternate food for the predator, harvesting of both species,” Int. *J. Open Problems Compt. Math.*, Vol.1¸ no.1.

[10] Lotka A.J., 1925, “Elements of Physical Biology,” *Williams & Wilking Baltimore*.

[11] Meyer W.J., 1985, “Concepts of Mathematical Modelling,” Mc. *Grawhill*.

[12] Phani Kumar N., Seshagiri Rao N., and Pattabhi Ramacharyulu N.Ch., 2009, “On the Stability of a Host – A Flourishing Commensal Species pair with Limited Resources,” *International Journal of Logic Based Intelligent Systems*, Vol.3, No.1.

[13] Shanker N., and Pattabhi Ramacharyulu N.Ch., “On the stability of a four species: A Prey-Predator-Host-Commensal-Mutual-Syn Eco-System-I,” *International eJournal of Mathematics and Engineering*, Vol.1, Issue 3, p. 304-316.

[14] Shanker N., Lakshmi Narayan K., and Pattabhi Ramacharyulu N.Ch., “On the stability of a four species: A Prey-Predator-Host-Commensal-Mutual-Syn Eco-System- II,” *International Journal of Mathematical Sciences and Engineering Applications*, Vol. 4, No. V, p. 409-427.

[15] Shanker N., Lakshmi Narayan K., and Pattabhi Ramacharyulu N.Ch., “On the stability of a four species: A Prey-Predator-Host-Commensal-Mutual-Syn Eco-System- III(Predator Washed out states),” *Global Journal of Pure and Applied Mathematics* (Communicated).

[16] Srinivas N. C., 1991, “Some Mathematical Aspects of Modellingin Bio-medical Sciences,” Ph.D thesis, Kakatiya University.

[17] Volterra V., 1931, “Leconssen La Theorie Mathematique De La Leitte Pou Lavie,” *Gauthier-Villars*, Paris.