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A GENERAL MATHEMATICAL MODEL OF ECOLOGICAL AMMENSALISM

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ABSTRACT

This paper aims mainly to develop the knowledge and creativity in Mathematical modeling. Discovery and brief history of mathematical modeling are elucidated. Classifications on Mathematical modeling are presented. Phases of mathematical modeling are explained. Ecology and its importance is highlighted. Various interactions are explicated with Mathematical significances. Moreover, Basic concepts and the formulation of Mathematical model of Ecological Ammensalism with multifarious resources are explicated. General Objectives of the study of a Mathematical model in view of Stability analysis are invented.

Keywords: Mathematical modeling, Ecology, interaction, stability analysis

AMS Classification: 92 D 25, 92 D 40

1) Introduction

Mathematical modeling is an important interdisciplinary activity, which involves the study of some aspects of diverse disciplines. Biology, Epidemiology, Physiology, Ecology, Immunology, Bio-Economics, Genetics, Pharmacokinetics are some of those disciplines. Mathematical Modeling is a quest of methodical scientific computing of the interaction between the multifarious species. Ever since its inception it has been extending its yeoman services for the development of science and technology in general and engineering in particular. It has become the backbone of modern scientific development. It has extended its sphere with manifold dimensions and every branch of mathematics has its own importance. This mathematical modeling has raised to the zenith in recent years and spread to all branches of life and drew the attention of every one.

Mathematical biosciences, also called Biomathematics is an interdisciplinary subject with a vast and exponentially growing literature spread over diverse disciplines for the last one century. Contributions to it have been made by mathematicians, physicists, computer scientists, ecologists, medical scientists, demographers and many others.In mathematical biosciences, we study the applications of mathematical modeling and mathematical techniques to get an insight into the problems of real life.To mention a few, it covers mathematical ecology, mathematical demography, mathematical bio-economics, mathematical medical sciences and mathematical agriculture and so on.

Mathematical modeling in mathematical life sciences is an attempt to identify and describe some instances of day-to-day life in the language of mathematics.It is an endeavor as old as the first human being and as modern as tomorrow's newspaper.While widening and deepening the scope of mathematical modeling in life and medical sciences, one is not just

restricted to the use of mathematical techniques already known.It is very interesting to note that one of the important roles of mathematicians working in the areas such as life, medical and social sciences is to evolve new mathematical techniques dealing with complex situations which would arise in nature and that too in our routine. Situations in life sciences are often quite complex. As such one should have some insight into a situation before attempting to formulate a new mathematical model.Once a model is formulated, its consequences can be noticed by using a suitable mathematical technique and the results are compared with the observations. The discrepancies between theoretical conclusions based on the model and real world observations suggest further improvements to the model.
Mathematical modeling thus essentially consists of translating real world problems, solving the mathematical problems and interpreting the solutions in the language of real world.

A real world problem in all its generality can seldom be translated in to a mathematical problem and even if it can be so translated, it may not be possible to solve the resulting mathematical problem in a satisfactory manner.Hence it would be necessary to 'simplify' or 'idealize' or 'approximate' the problem by another problem, taken as a replica to the original problem and at the same time that can be meaningfully translated and solved mathematically. In this idealization process, all the essential features of the problem would be retained, giving up those features which are not inevitable nor so relevant to the situation under investigation.

If no satisfactory comparison is noted between the conclusions based on the model and observations in the real world situation, the assumptions in model-making are either modified or another form / structure for the mathematical model is fabricated.A variety of mathematical techniques are being employed in analyzing the model equations. This process would continue till a satisfactory model close to the real situation is formulated. The mathematical and biological models complement each other in the process of understanding the reality.

The mathematical model, structured on a biological base, would enable in the prediction/quantitative estimation of population strength at subsequent instants of time. In the absence of a biological model, the mathematical treatment would become more and more abstract and in general, making the whole treatment difficult to analyze in drawing meaningful conclusions.The absence of mathematical treatment makes difficult to identify the general relevance of a particular biological model.

Certain principles are to be followed to formulate a mathematical model. The diverse techniques adopted in mathematical modeling have been illustrated somewhat exhaustively by Kapur[18] in his treatise on mathematical modeling.Another detailed monograph by Kapur[19] deals with diverse topics on mathematical modeling in biological and medical sciences. These two works provide a vast information and the basic ideas of modeling in a wide spectrum of areas of knowledge and function as source of material at the initial stages of the investigations in the areas of knowledge of biomathematics.

Modeling may thus be construed as a science dealing with the interaction between mathematics and other subjects - an academic discipline concerning some aspects of the everyday world. The interaction process can be viewed as consisting of the following phases.

2. Classifications Of Mathematical Modeling

A Mathematical model uses mathematical languages to describe a system. These models are applied not only in natural and technical sciences but also in social sciences. The whole process of formulating a mathematical model is named as "Mathematical Modeling". According to the definition of Eykhoff a mathematical model is "A Representation of the essential aspects of an existing system which presents knowledge of that system in a usable form". These Mathematical models take many forms. Even though they include dynamical systems statistical models defines equations or game theoretic models but they are not limited to them. Along with other types of models these mathematical models are prone to overlap with a given model involving a variety of abstract structures.This Mathematical model is used by Mathematicians to analyze a system that is to be controlled or optimized. To have a comprehensive understanding of Mathematical models they can be classified as follows.

i) Linear Vs. Non-linear.
ii) Deterministic Vs probabilistic
iii) Static vs Dynamic
iv) Lumped vs distributed parameters.

i) Linear verses Non-Linear: It is a known fact that mathematical models are generally composed by variables. These variables are abstractions of quantities of interest in described systems and operators which act on these variables.Those operators can be algebraic operators, functions, differential operators etc.If all the operators which are in a mathematical model exhibit linearity that Mathematical model is defined as linear. If it is not so that model is considered as a non-linear matheamatical model.

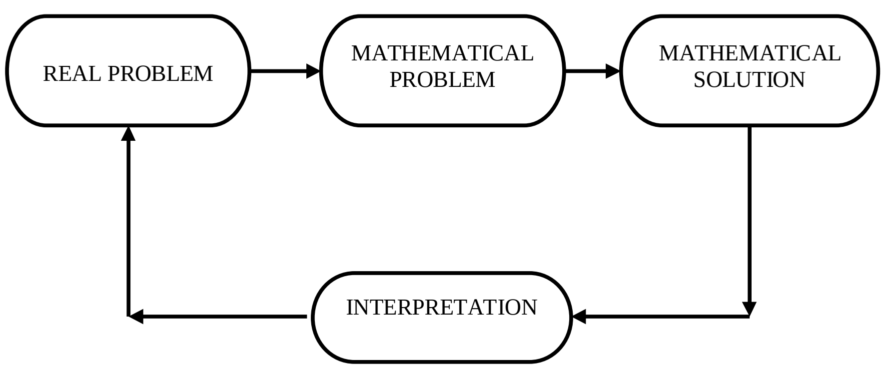
ii) Deterministic verses probabilistic (Stochastic): In the Mathematical model where every set of variable states is uniquely determined by parameter and also by sets of previous states of these variables, that model is known as a deterministic Mathematical model. According to this definition deterministic models perform the same way for a given set of initial conditions. In a stochastic model the variable states are not described by unique values.They are described by probability distribution. Randomness is found in this stochastic model.

iii) Static verses Dynamic: The model which does not account for the element of time is known as static model. But the dynamic model accounts for the element of time.That is why dynamic models are represented with differential equations.

iv) Lumped verses distributed parameters: Lumped Mathematical model is homogeneous. Throughout the entire system its state will be consistent. Whereas the model of distributed parameters is heterogeneous. In this system the parameters are distributed. There will be varying state within the system.The distributed parameters are represented with partial differential equations.

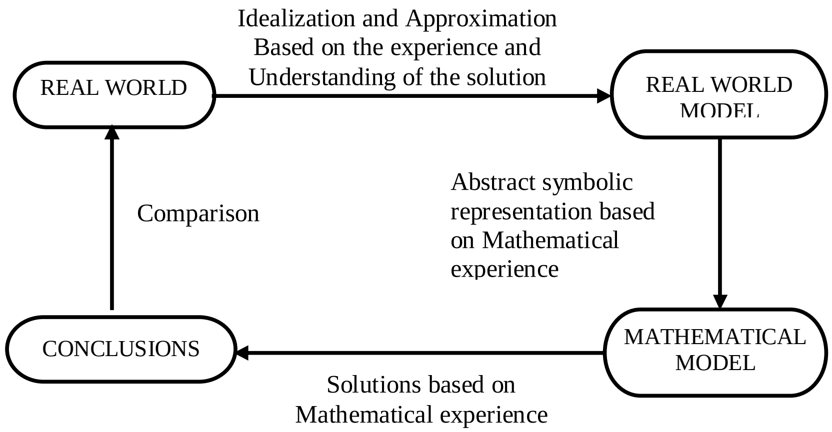
Mathematical modeling transliterates real world problems using Mathematical alphabets.After that this modeling solves the mathematical problems and interprets the solutions in the languages of the real world.This system can be illustrated as follows.

Basic block diagram of real world problems can be shown as below:



It is evident that to get a suitable solution a real world problem is to be translated into a Mathematical problem. It is also necessary to simplify the problem by another problem which stands very close to the original problem. In this method of idealization we try our best to retain all the essential features of the problem.At the same time we give up those features

which are not so relevant to our investigation.In this light of modification we can change the above diagram as follows.
Modified Diagram of real world problems can be illustrated as below:



In this process of Mathematical modeling if we do not get satisfactory comparisons between the conclusions that are based on the model and the real world problems then we modify the assumptions. We try for another structure for Mathematical model.In the field of analyzing the model equations, different types of mathematical techniques are put into practice. In order to understand the reality, these Mathematical and biological models help each other.If the Mathematical model is structured in biological model it will enable us in the quantitative estimation of population strength in the fourth coming moments of time.If we do not have a biological model the mathematical treatment becomes more hypothetical.In that situation it becomes difficult to draw the meaningful conclusions based on our analization.Kapoor mentioned this in his treatise on mathematical modeling exhaustively about certain principles that are to be followed to formulate the Mathematical model.He has also illustrated the diversed techniques that can be adopted in Mathematical model.Kapoor presented a full fledged monograph with different topics on mathematical modeling in biological and medical sciences. The said two references of Kapoor give us vast information along with the basic ideas of modeling in diverse areas.At the initial stages these references act as the source of the investigations in the field of bio-mathematics.

3. Phases of Mathematical Modeling:

i) Identification of the problem in the scientific, in particular, biological / medical / social setting.
ii) Formulation of the mathematical model.
iii) Solution of the mathematical problems that would arise in the study of the model selected.
iv) Developments of algorithms and associated computer programs for the relevant computations.
v) Explanation and interpretation of the results in the context of the original problem and the communication of this information to all those interested. This amounts to the evaluation of the results.

Mathematical models have become important tools in biological investigations with an iterative procedure of information collection.If such models are properly developed and used, they can provide insight into the relations between the physical variables and process influencing the system being studied.The resulting interplay between the experimental investigation and the theoretical model can be an essential factor in designing experiments and in the interpretation of data.

In 1202, Lenardo of Pisa [22] (also known as Fibonacci) proposed a population model for rabbit growth, employing suitable arithmetic, to estimate the number of pairs of rabbits after each successive month having started with one pair reproducing on a particular pattern. A more serious attempt in formulating problems of life sciences was made by Giovanni Borelli [15]. Arey Went Worth Thompson [4] published his work on growth and form that marks the beginning of modern theoretical biology employing mathematics as an essential tool.

Research in Theoretical ecology was initiated by Lotka[23].The concept of mathematical modeling can be had from the book written by Meyer [28].Detailed information on modeling of infectious and epidemic diseases are noted in a monograph written by Bailey [5]. The lecture material of Frauenthal [11] provides information on modeling on epidemic diseases such as helminthes infections etc.Modeling of communicable diseases like gonorrhea, in the population is given in a monograph written by Hethcote and Yorke [17]. Some more information on modeling can be referred in monographs written by Brams et al [7], Heberman [16], Lucas et al [24].Modeling in life and Social sciences with the help of ordinary differential equations has been discussed in treatise of Braun [8]. Mathematical modeling in Immunology has been discussed by Marchuck [26].

In general, there are two types of mathematical models: Deterministic and Stochastic. In deterministic models, the formulation of the systems depends upon various axioms / assumptions to be considered due to the related system-biology and these may be presented in the form of autonomous or non-autonomous, ordinary / partial (linear or nonlinear) differential or integro-differential equations or difference equations, etc.The stochastic models are in general, probabilistic models involving difference or differential-difference equations, etc.

4. Ecological Models:

Ecology deals with the study of living beings such as animals and plants in relation to their habits and habitats.It mainly deals with the evolutionary biology which explains us about how the living being is regulated in nature.Ecology being a science, synthetic in principle, concerns with a wide variety of models.It is the only science that needs minimum time and labour for its introduction to a layman.The mathematical methods are widely exploited for solving ecological problems of diverse nature. One of the several problems central to ecology, in general is that of ecosystem stability "steadiness".It is clear that only a stable ecosystem may exist over a long time. On the other hand, the limits of stability determine those maximal ecosystem loads, the exceeding of which will lead to an "ecological catastrophe", i.e., destruction of ecosystem.The problem of stability always presents itself, when we exploit natural populations and communities, evaluate the environment population limits, take into account the implication of certain projects in management of natural resources, or perform appropriate feasibility studies.All these estimates are clear and convincing only if they are quantitative. Thus we need model formulation mathematically and fabricate suitable mathematical methods if the already known methods are found to be inadequate or unsuitable in evolving the criteria for the stability of the system.

A biological community, which exists for sufficiently long time in a more or less invariable state, should possess intrinsic abilities to resist perturbations coming in abundance from the environment.This ability of an ecosystem is usually termed system stability. A community is considered to be stable, if the number of member species remains constant over sufficiently long time intervals.On the other hand, an advanced theory of mathematical stability is available (with a host of applications in science and engineering) which deals not with real objects, but with their mathematical models.Therefore if we have a good model of an ecosystem (in terms of differential or difference equations), then the stability of real community can be deduced from our model by conventional methods of stability theory. Stability in general refers to some solutions of a system of equations. For instance a community or ecosystem may be believed to be stable, when its model trajectories in the phase space stay within a given bounded domain for a sufficiently wide scope of perturbation.

5. Various Ecological Interactions between the two Species:

The nature of the interactions between species ranges from mutually beneficial state to mutually harmful state. Some of the frequently noticed species interactions are listed hereunder. These interactions are however not always static.

(i) **Neutralism**: Each species does not have any influence on the other. True neutralism, a rare/remote situation, is impossible to establish as all species in nature interact in some manner with each other. The insignificantly small interactions are normally referred as **neutralism**.

(ii) **Ammensalism**: Ammensalism a type of symbiosis is the interaction between two species where one species harms the other without getting effected due to the interaction. In this process the behavior of one organism is detrimental to the growth of another organism. The harmed species is called the **Ammensal** and the harming one the **Enemy**. A common example for this is the bread mould pencillium.

(iii) **Commensalism**: In commensalism one organism benefits the other without getting effected due to the interaction i.e It is neither benefited nor harmed. The beneficial species is the **Commensal** while the other benefiting species the **Host**. Remora living with a Shark is an example of this commensalism.

(iv) **Competition**: Generally the word competition means the struggle to come over the other participants for existence.The interaction between the individuals who are mutually detrimental to each other is known as competition. When two or more organisms try to live in the same community with the same resources and in the same environment, they compete with one another for their survival. This would arise in struggle for existence whenever the resources are limited.

(v) **Mutualism**: When the association between pairs of species resuls in mutual benefits that interaction is known as mutualism.These interactions of mutualism are of two kinds - (i) Symbiotic (ii) Non-symbiotic.In the mutual interaction where the mutual individuals interact physically and their relationship is biologically essential for survival, it is known as symbiotic Mutualism. In Mutualism, interaction where the mutualists live independent lives apart from the fact that they cannot survive without each other is known as non-symbiotic mutualism.

(vi) **Predation** : Predation is the interaction where one species gets benefits at the expense of the other.In this interaction between two organisms one organism captures bio-mass from another. This is called as predation. In this interaction one organism eats away the another with its closeness of association.

These interactions between two species(X and Y) are shown in a table given as below:

S. No.	Effect on X	Effect on Y	Type of interaction	Species-Names
1	0	0	Neutralism	X and Y are Neutrals
2	-	0	Amensalism	X:Ammensal, Y:Enemy
3	+	0	Commensalism	X:Commensal,Y:Host
4	-	-	Competition	X and Y are Competitors
5	+	+	Mutualism	X and Y are Co-operators
6	+	-	Predation	X:Predator, Y:Prey

In the above table some relationships are shown according to the effect they have on each partner. "0" is no effect,"-" is detrimental and "+" is beneficial.

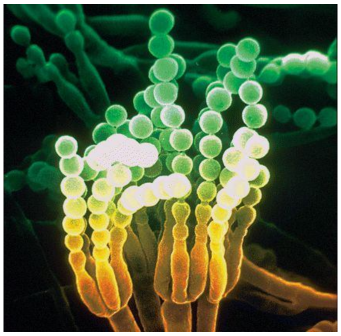
The present thesis deals with some problems related to Ammensalism in Ecosystems.

6. Ecological Ammensalism

In the list of species interactions, Ammensalism occupies peculiar yet important place.Ammensalism with all its manifold dimensions is rendering its remarkable services in the field of ecology.As it is stated earlier Ammensalism is an ecological relationship between two species where one species (S_2) adversely effects the other species (S_1) without itself being effected in any way. The adversely effected Species (S_1) is referred as **Ammensal** species and the other, the harming one(S_2),the **Enemy/ Harming** species. A few examples of Ammensalism among some species are reported below.

Examples of Ammensalism:

a) **Pencillium** secretes and kills bacteria. The **pencillium** does not get any benefits by killing the bacteria.

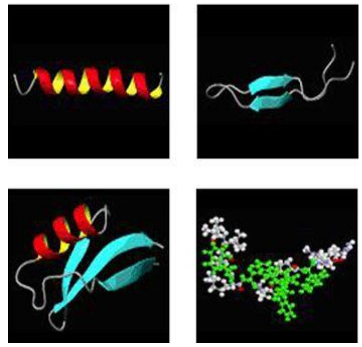


Pencillium

b) **Algal** blooms can lead to the death of many species of fish, but the **Alga** do not get any benefit from the destructions of those individuals.

Algal blooms

c) Another example of **Ammensalism** is that in which one organism releases a specific compound that causes harm to the another organism. They can be listed as below.
i) Antibiotics (ii) Bacteriocins (iii) Antibacterial peptides (iv) Acidic fermentation products.



Antibacterial peptides

d) **Black walnut tree:** A chemical product is released from **Black walnut tree** - roots which causes harm to the plants around it along with numerous other species.

Black walnut tree

e) Ammensal relationships are also found in **human beings**. Several human activities are detrimental to the other species. Moving in automobiles results in air pollution leading to health problems of other living beings .Damage is caused to lichens and plants by the electricity generating stations.In such cases some species are damaged but the human beings receive no direct benefit.

f) When **forests** are destroyed to provide wood for industrial purposes, many birds, mammals and other animals that live in the forest will suffer a huge -habitat loss. Even though we see some economic benefits to the human beings in this act, there are no particular benefits to the people from this damage.

g) **Pine tree** stunts the growth of the vegetation under it ,as the vegetation is deprived of the sun light that is cut off by the large canopy of the trees.

Pine tree

The model equations for each of the interactions mentioned above have been presented in the treatises of authors such as Kapur [18], Addicott, J.F. [1], Boucher [6],Svirezhev and Logofet [41], Freedman [12], Pielou, E.C. [32]and Cushing [10]. The concepts of Gauss,G.F [13] , George F. Simmons[14] and Ogata[30] helped to develop the field of Mathematical Modeling. May [27], Rogers et al [33], Smith [39], Paul Colinvaux [31] and Volterra[44] etc. based on the ideas suggested therein, the fundamental equations for a general Ammensal interaction between two species S_1(Ammensal) and S_2(Enemy) can be modeled as a pair of nonlinear coupled ordinary differential equations with the help of the following block diagrams.

(i) Equation for the growth rate of Ammensal species (S_1):

Rate of change of Ammensal species(S_1)	=	Natural growth rate of Ammensal species (S_1)
	−	The growth rate reduction of Ammensal species (S_1) due to limitations of its natural resources
	−	The growth rate reduction of Ammensal species (S_1) due to competitive inhibition from the Enemy species (S_2)
	+	Replenish rate

(ii) Equation for the growth rate of Enemy species (S_2):

Rate of change of Enemy species(S_2)	=	Natural growth rate of Enemy species(S_2)
	−	The growth rate reduction of Enemy species (S_2) due to limitations of its natural resources
	+	Replenish rate

Note :(1) the natural growth rate is in accordance with the Malthus Law[25](1798)

(2) The growth rate reduction due to limitation of the corresponding natural resources is in accordance with the Verhulst[43](1837)

7. The Basic model equations :

$$\frac{dN_1(t)}{dt} = a_1 N_1(t) - a_{12} N_1(t) N_2(t) - a_{11} N_1^2(t) + H_1(t) \qquad (7.1)$$

$$\frac{dN_2(t)}{dt} = a_2 N_2(t) - a_{22} N_2^2(t) + H_2(t) \qquad (7.2)$$

with the conditions $N_i(0) = N_{i0} \geq 0, i = 1,2$;

NOTATIONS ADOPTED:

N_1 (t) : The population of the species S_1 at time t

N_2 (t) : The population of the species S_2 at time t

a_i : The natural growth rates of S_i, i = 1, 2.

a_{ii} : The rate of decrease of S_i; due to its own insufficient resources ,i=1,2.

a_{12} : The inhibition coefficient of S_1 due to S_2 i.e The Ammensal coefficient.

a_{21} : The inhibition coefficient of S_2 due to S_1

$H_1(t)$: The replenishment or renewal of S_1 per unit time

$H_2(t)$: The replenishment or renewal of S_2 per unit time

K_i : a_i/a_{ii} are the carrying capacity of N_i, i = 1, 2.

α : a_{12}/a_{11} is the coefficient of Ammensalism.

h_1 : $a_{11} H_1$ is rate of harvest of the Ammensal

h_2 : $a_{22} H_2$ is rate of harvest of the enemy.

The state variables N_1 and N_2 as well as the model parameters a_1, a_2, a_{11}, a_{22}, K_1, K_2, α , h_1, h_2 are assumed to be non-negative constants.

The non-linearity of the set (7.1 and 7.2) makes the equation intractable to obtain an exact analytical solution enabling a thorough investigation by the mathematical scientists / bio-mathematicians. However, one can examine the character of the solution in some special cases. Adopting a method such as space-portrait analysis, the behaviour of the species S_1 and S_2 in cases like Competition, Prey-Predation etc. has been dealt extensively within the treatises of Braun[8], Freedman [12], Kapur[19] and Varma [42]. More work on prey-predation and competition interactions can be seen in Rish and Boucher [38] and May [28].

A general discussion on multi species population models can be found in Bronstein[9], J.L.Kapur [18]. Recently Srinivas,N.C [40] discussed some mathematical aspects of modeling in Bio-Medical Sciences, which was later followed by Lakshmi Narayan [20,21] who has dealt with some threshold theorems for a prey-predator model with harvesting. More recently Archana Reddy[2,3],Rama Sharma,B.B [34,35] examined a two species competitive Eco-system with decay and replenishment for one of the species, while Ravindra Reddy investigated mutualism between two species[36,37].

In stability analysis of the equilibrium states of some Ammensal ecosystem models has been carried out to arrive at the criteria for
(i) Local (ii) Global stabilities.

8. General Objectives of the study in Ecological Ammensalism:

- To develop few mathematical models for ecological Ammensalism between two or three interacting species with multifarious resources.
- To identify all the equilibrium points of the models and carry out the stability criteria analysis for existing equilibrium states.
- To illustrate the trajectories of perturbed species.
- To establish the global Stability of the models by constructing a suitable Liapunov's function in some special cases.
- To explicate various interactions between the Ammensal and enemy species by analytical or numerical mathematical study
- To draw some threshold diagrams where ever necessary and feasible.

9. Concepts of Equilibrium States and their Stability:

The basic equations of two species with Ammensalism can be written as of first order non-linear coupled ordinary differential equations.

$$\frac{dN_1}{dt} = f_1(N_1, N_2) \tag{9.1}$$

$$\frac{dN_2}{dt} = f_2(N_1, N_2) \tag{9.2}$$

when N_1 is the population of Ammensal species, N_2 is the population of enemy species and f_1, f_2 are functions of N_1& N_2

The equilibrium state of the above system is obtained by

$$\frac{dN_1}{dt} = 0; \frac{dN_2}{dt} = 0 \quad \text{i.e., } f_1(N_1, N_2) = 0,\ f_2(N_1, N_2) = 0 \tag{9.3}$$

Let the roots be $\overline{N}_1, \overline{N}_2$ (9.4)

The equilibrium state is denoted by $(\overline{N}_1, \overline{N}_2)$, An equilibrium state $(\overline{N}_1, \overline{N}_2)$ is called

(i) Fully washed out state if $\overline{N}_1 = 0;\ \overline{N}_2 = 0$

(ii) Ammensal species washed out state if $\overline{N}_1 = 0;\ \overline{N}_2 \neq 0$

(iii) Enemy species washed out state if $\overline{N}_1 \neq 0;\ \overline{N}_2 = 0$

(iv) Co-existent state if $\overline{N}_1 \neq 0;\ \overline{N}_2 \neq 0$

10. Stability Analysis of Steady State:

To investigate the stability of a equilibrium state ($\overline{N}_1, \overline{N}_2$);

we consider $N_1 = \overline{N}_1 + U_2$, $N_2 = \overline{N}_2 + U_2$. (10.1)

where U_1 and U_2 are small perturbations over that equilibrium state.
After linearization, (neglecting higher order terms in U_1 and U_2)

we obtain $$\frac{dU}{dt} = AU \qquad (10.2)$$

$$\text{where } A = \begin{bmatrix} \frac{\partial f_1}{\partial \overline{\overline{N}}_1} & \frac{\partial f_1}{\partial \overline{\overline{N}}_2} \\ \frac{\partial f_2}{\partial \overline{\overline{N}}_1} & \frac{\partial f_2}{\partial \overline{\overline{N}}_2} \end{bmatrix} \; and\, U = \begin{bmatrix} U_1 \\ U_2 \end{bmatrix} \qquad (10.3)$$

The solution is of the form

$$\begin{bmatrix} U_1 \\ U_2 \end{bmatrix} = \begin{bmatrix} A_1 \\ A_2 \end{bmatrix} e^{\lambda t} \qquad (10.4)$$

where λ satisfies the secular equation

$$\text{i.e., } \lambda_2 - \lambda\left(\frac{\partial f_1}{\partial \overline{\overline{N}}_1} + \frac{\partial f_2}{\partial \overline{\overline{N}}_2}\right) + \left(\frac{\partial f_1}{\partial \overline{\overline{N}}_1}\frac{\partial f_2}{\partial \overline{\overline{N}}_2} - \frac{\partial f_1}{\partial \overline{\overline{N}}_2}\frac{\partial f_2}{\partial \overline{\overline{N}}_1}\right) = 0 \qquad (10.5)$$

The equilibrium state ($\overline{N}_1, \overline{N}_2$) is stable if both the roots of the equation (10.5) are both negative when they are real and have negative real parts in case of complex roots.

11. Liapunov's method of Global Stability:

If the total energy of a physical system has a local minimum at a certain equilibrium point then the point is said to be stable. Liapunov generalized this principle by constructing a function $E(N_1,N_2)$ whose rate of change is given by

$$\frac{dE}{dt} = \frac{\partial E}{\partial N_1}\frac{dN_1}{dt} + \frac{\partial E}{\partial N_2}\frac{dN_2}{dt} = \frac{\partial E}{\partial N_1} f_1 + \frac{\partial E}{\partial N_2} f_2 \qquad (11.1)$$

Liapunov's Function: A positive definite function $E(N_1,N_2)$ with the property that $\frac{dE}{dt} = \frac{\partial E}{\partial N_1} f_1 + \frac{\partial E}{\partial N_2} f_2$ is negative semi-definite is called a Liapunov's function for the system (9.1) and (9.2)

The following is the theorem that characterizes the global stability.

Theorem (A) : If there exists a Liapunov's function $E(N_1,N_2)$ for the system (9.1) the equilibrium point (0,0) is stable. Further if this function has additional property that $\frac{dE}{dt}$ is negative definite then (0, 0) is asymptotically stable. The following theorem helps to ascertain definiteness of a Liapunov's function:

Theorem (B): The function $E(x,y) = a x^2 + b x y + c y^2$ is positive definite if a>0 and $b^2 - 4ac < 0$ and negative definite if $a<0, b^2-4ac<0$.

Simple Equilibrium points of Non-linear Systems Linearization:

Consider an autonomous system (9.1) and (9.2)

$\frac{dN_1}{dt} = f_1(N_1,N_2)$ and $\frac{dN_2}{dt} = f_2(N_1,N_2)$ with an equilibrium point (0,0) .If f_1 and f_2 can be expanded in power series in N_1,N_2 then the above system takes the form

$$\left.\begin{aligned}\frac{dN_1}{dt}&=a_1N_1+b_1N_2+c_1N_1^2+d_1N_1N_2+e_1N_2^2+......\\ \frac{dN_2}{dt}&=a_2N_1+b_2N_2+c_2N_1^2+d_2N_1N_2+e_2N_2^2+......\end{aligned}\right\} \quad (11.2)$$

when $|N_1|$ and $|N_2|$ are small, their second and higher degree terms can be discarded from (9.1) and (9.2) and the linearised form of the system would be

$$\frac{dN_1}{dt}=a_1N_1+b_1N_2 \quad (11.3)$$

$$\frac{dN_2}{dt}=a_2N_1+b_2N_2 \quad (11.4)$$

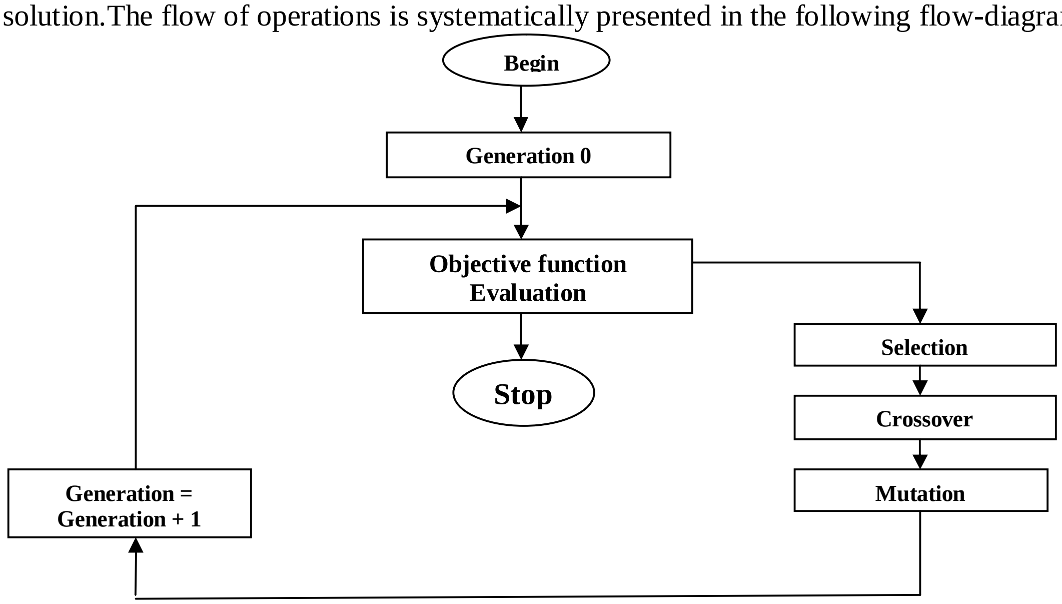
An equivalence of the stability of equlibrium points of these two systems (9.1; 9.2) and (11.3; 11.4) can be realized by the following theorem on asymptotic stability.

Theorem(C): Let (0,0) be equlibrium point of non-linear system (9.1) and (9.2). Consider the related linear system (11.3) and (11.4). If the point (0,0) is asymptutically stable for the later system (11.3) and (11.4) then (0,0) is also so for the first system (9.1) and (9.2).

12. Genetic Algorithm:

As the name suggests, Genetic algorithm (GA) follows the principle of genetics which is popularly known as survival of the fittest. In optimization objective function value depends on a set of decision variables. In GA each decision variable is coded as chromosome and a set of chromosomes (decision variables) gives rise to a gene. A particular number of such randomly picked populations is known as population size. The objective function estimated for a given genetics determines its fitness. The individiuals in the population are randomly selected, recombined, mutated, and either eliminated or retained according to their relative levels of fitness.

A simple genetic algorithm comprises of three operators; reproduction, crossover, and mutation. Each individual of the randomly chosen initial population is evaluated according to a fitness function (objective function). Then, the reproduction operation is performed by choosing individuals according to their relative fitness. High-fitness individuals are selected in a greater number of times, in proportion to their relative fitness. Here it is noteworthy that reproduction alone can not produce any new (different) individuals into a population. The crossover and mutation operations are performed to new individuals to complete one generation of evolution. Several such generations are carried out to obtain the optimal solution.The flow of operations is systematically presented in the following flow-diagram.



The following parameters of GA are found to be satisfactory for the present study.

Number of variables	Problem specific.
Maximum population size	50
Probability of cross over	0.8
Probability of mutation	0.06
Random seed	0.0625
Distribution index for cross over	5
Distribution index for mutation	80

Real coded Genetic algorithm for a single objective optimization, which is made available through KANGAL laboratories.

13. Numerical Approch

The mathematical model which is in the form of a pair of non-linear coupled ordinary differential equations can also be solved by using R-K fourth order method/Picard's method with a step length of 1e-3; R-K method write up. Mat lab graphic tools can be employed for constructing the structures. 0-5 range may be taken for the parameters of the model $a_{1,}a_{11,}a_{2,}$ $a_{22,}a_{12,}$ $N_{10,}$ and $N_{20.}$ Numerical study can be focused on tracing the interaction between the Ammensal- enemy species at possible values of the dominance- reversal time (t^*). In general, the nature of dominance- reversal time is identified by fixing few parameters. But in few investigations, authors may concentrate on finding every possible solution for deriving the interaction between Ammensal species. The initial values of N_{10} and N_{20} for solving the model are automatically to be picked up by the Genetic algorithm while tracing the possible solutions in entire search space. The necessary graphs are to be illustrated for the representative solutions.

14. Open problems

Ecological situations such as the following are under consideration for further investigation.

1) (a)An ecosystem consisting of a prey ,a predator with unlimited resources and a enemy Ammensal to the prey with resources (i) unlimited and (ii) limited and also with harvesting of one or other species.

(b)An ecosystem consisting of a predator that survives on an immigrating prey that is also an Ammensal to an enemy. It is also relevant to examine the delayed effect in the above mentioned problems.

(c) A three species ecosystem consisting of a prey with limited resources, a harvested predator with unlimited resources and a harvested enemy Ammensal to the prey, the effect of delay(s) of the interaction in the above cases.

2) (a) A model of four species (S_1, S_2, S_3, and S_4); S_1 and S_2 are neutral to each other, S_3 is a predator living up on S_1, S_4 is a predator living up on S_2 and S_3 &S_4 are competitive struggling for existence.

(b) A model of four species (S_1, S_2, S_3, and S_4); S_1 and S_2 are neutral to each other, S_3 is a predator living up on S_1, S_4 is a predator living up on S_2 and S_3 &S_4 are co-operate with each other struggling for existence and so on.

In these investigations, the following items can also be taken up.

(i)Harvesting of one or other of the species.

(ii)The effect of time delay of one or other interactions.

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