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A SERIES SOLUTION OF MOISTURE CONTENT IN VERTICAL GROUNDWATER FLOW THROUGH UNSATURATED HETEROGENEOUS POROUS MEDIA

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Abstract
The study is intended to present the theoretical model of moisture content in vertical groundwater flow through unsaturated heterogeneous porous media with aqueous diffusivity coefficient. The governing non-linear partial differential equation of vertical flow through an unsaturated soil is solved analytically using the Richards's equation in which governing equation converted into ordinary differential equation by using similarity transformation together with boundary conditions. The solution is obtained in term of ascending Maclaurin's power series which represents the moisture content $\theta(z,t)$ for any depth $z,t>0$ and its graphical presentation is given by using MATLAB coding.

Key words: Unsaturated porous media, aqueous diffusivity, similarity transformation
AMS (Subject) Classification 2010: 76T99, 76S05, 76M99

1. Introduction

In nature, groundwater is a key element in many geological and hydro-chemical processes, geotechnical factor conditioning soil and rock behaviour and component of ecological system which sustains spring discharge, river base flow, lakes and wetlands. The soil plays one of the most important roles in the hydrological cycle. It is a three-phase porous medium, where the phases are particles of ground rock or clay, water and a water vapour air combination. The unsaturated zone is the part of the subsurface between the land surface and the groundwater table. The definition of an unsaturated zone is that the water content is below saturation (for the specific soil). Hence, 'unsaturated' means that the pore spaces between the soil grain particles or the pore space in cracks and fissures are partially filled with water, partially with air. The unsaturated zone can be from meters to hundred of meters deep.

In many practical situations, the flow of water through soils is unsteady and slightly saturated due to the moisture content changes as a function of depth and time. It is slightly saturated because all the pore spaces are not completely filled with flowing liquid. The phenomenon of the one dimensional vertical groundwater flow through unsaturated porous medium is of great importance for hydrologist, meteorologist, agriculturists and for people related with water resources sciences. The key benefit of this research is improved conceptual models of how all contaminants migrate through heterogeneous and variably-saturated porous media. Research activities are driven by the hypothesis that the reactivity of variably saturated porous media is dependent on the moisture content of the medium and can be represented by a relatively simple function applicable over a range of scales, contaminants and media [7].

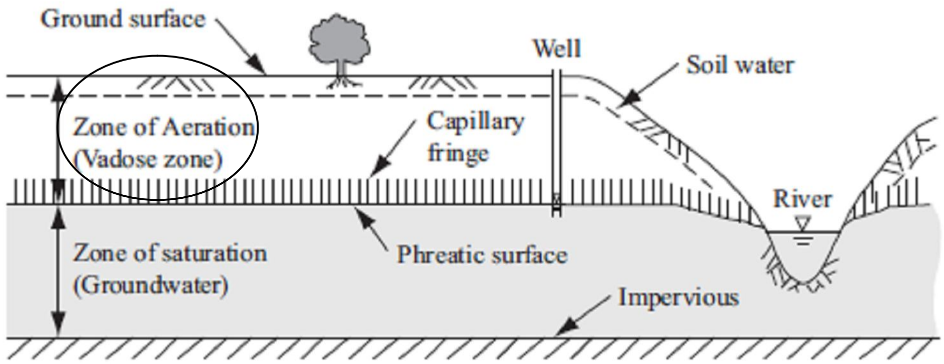

Figure 1: Representation of moisture zone [1].

The hydrological situation of such problem is confirmed by Verma [16]. Very few researchers have worked on this phenomenon in cracked and heterogeneous porous media, in which vast majority of the work has been done on homogeneous porous media. Here are some of the examples- Swartzendruber uses Philip's methods [11] to get graphical illustration of the mathematical solution for horizontal water function. Mehta [9] obtained an approximate solution by the method of singular perturbation technique. He considered the average diffusivity coefficient of the whole range of moisture content and treated as small constant. Ross [13] discussed the efficient numerical methods for infiltration using Richard's equation. Hari Prasad et.al. [4] had developed a numerical model to simulate moisture flow through unsaturated zones using the finite element method. This model is also applied to predict moisture contents during a field internal drainage test. Faybishenko [3] has given review of the theoretical concepts, has presented the results, and provided perspectives on investigations of flow and transport in unsaturated heterogeneous soils and fractured rock, using the methods of nonlinear dynamics and deterministic chaos. Meher and Mehta [7] have discussed Adomian decomposition method for moisture content in one dimensional fluid flow through unsaturated porous media with small diffusivity coefficients. Joshi et.al. [6] have obtained solution of one-dimensional and unsaturated fluid flow through porous media by group theoretical approach.

In the presented model, it is considered that the groundwater recharge takes place over the large basin taken as heterogeneous porous medium. In this case the soil properties like as porosity and permeability may be vary from one place to another place and the flow may be assumed vertically downwards through slightly saturated porous media, i.e. initial saturation is non-zero. In the present analysis the aqueous diffusivity coefficient is taken as directly proportional to the moisture content [8]. An approximate solution for the problem of vertical groundwater in slightly saturated heterogeneous porous media has been obtained in term of ascending Maclaurin's power series using the similarity transformation.

2. Statement of the problem

In the investigated model it is considered that the groundwater recharge takes place over the large basin of such geological location shown in figure 2 that the sides are limited by rigid boundaries and the bottom by a thick layer of water table fig. 2. In this case the flow takes place vertically downwards neglecting spreading in other directions (i.e. small amount of water may spread in other directions but it is very small compared to large size basin) through slightly saturated porous media. The aqueous diffusivity coefficient is taken as directly proportional to moisture content.

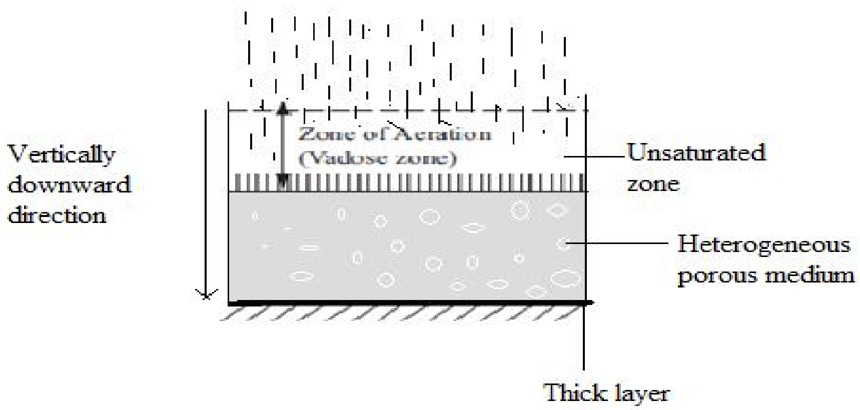

Figure 2: Moisture content in unsaturated zone

The assumptions have been considered for present analysis such as the medium is heterogeneous and there is no air resistance to the flow (i.e. the porous medium contains only the flowing liquid water and empty voids). The air in the void space is approximately at atmospheric pressure i.e. air is stationary; however the porosity and permeability are taken to be variable. The flowing liquid (water) is considered continuous at a microscopic level, incompressible and isothermal, where the moisture content at the soil surface is constant and near saturation or rainfall or irrigation rate is constant and last assumption is Darcy's law is applicable [4].

3. Mathematical formulation

In our assumed model, when water or rainfall takes place in downward direction at soil surface $z = 0$, it satisfies Darcy's law [2] for the motion of water in porous medium, we get

$$v = -K \nabla h$$

(1)

where 'v' represents the volume flux of moisture, 'K' indicates the coefficient of aqueous variable conductivity and '∇h' is the gradient of the whole moisture potential.

When such flow takes place, it must satisfies the equation of continuity [12] for unsaturated porous media is given by

$$\frac{\partial}{\partial t}\left(\rho_s \phi S\right) = -\nabla M$$

(2)

where 'ρ_s' is the bulk density of the heterogeneous medium on dry weight basis, '$M = \rho v$' represents the mass of flux of the water at any time $t \geq 0$, 'ρ' is flux density, 'S' is saturation of water and 'ϕ' is variable porosity of heterogeneous porous media.

We consider the laws of variation in the porosity and permeability of the uniform heterogeneous medium is defined as function of z as Oroveanu [10].

$$\phi = \phi\left(z\right) = \frac{1}{a - bz}$$

(3)

$$K = K\left(z\right) = K_c\left(1 + a_1 z\right)$$

(4)

Where a, b and a_1 are constants.

Using the incompressible of water and considering the fact that the water content or moisture content $\theta = \phi\left(z\right) S\left(z,t\right)$ [5], where ϕ a variable porosity is in heterogeneous porous medium and S is saturation of water which is a function of z and t.

Hence using above relation, equation (2) reduce to

$$\frac{\partial}{\partial t}\left(\rho_s \theta\right) = -\nabla M = -\nabla\left(\rho v\right)$$

(5)

Combining equation (1) and (5), we get

$$\frac{\partial}{\partial t}\left(\rho_s \theta\right) = \nabla\left(\rho K \nabla h\right)$$

(6)

As in the present problem, it is considered that the flow takes place only in vertical downward direction then $\Delta h = dh/dz$ [14], equation (6) reduced to,

$$\rho_s \frac{\partial \theta}{\partial t} = \rho \frac{\partial}{\partial z}\left(K \frac{\partial h}{\partial z} \right)$$

(7)

For more simplification considering the relation, $h = \psi + gz$ [5], where g is gravitational constant, z is direction of flow and ψ is the capillary pressure potential, in equation (7), we get,

$$\frac{\partial \theta}{\partial t} = \frac{\partial}{\partial z}\left(\frac{\rho}{\rho_s} K \frac{\partial \psi}{\partial z} + \frac{\rho}{\rho_s} Kg \right)$$

(8)

This equation is known as Richards's equation.

It is well known that the capillary pressure potential ψ an moisture content θ are related by single valued function [5], so equation (8) will be,

$$\frac{\partial \theta}{\partial t} = \frac{\partial}{\partial z}\left(D(\theta) \frac{\partial \theta}{\partial z} + \frac{\rho}{\rho_s} Kg \right)$$

(9)

where, $D(\theta) = \frac{\rho}{\rho_s} K \frac{\partial \psi}{\partial \theta}$ which is the aqueous diffusivity coefficients. It is directly proportional to moisture content [8].

The equation (9) reduce to

$$\frac{\partial \theta}{\partial t} = \alpha_0 \frac{\partial}{\partial z}\left(\theta \frac{\partial \theta}{\partial z} \right) + \frac{\rho g}{\rho_s} \frac{\partial K}{\partial z}$$

(10)

where α_0 is constant of proportionality.

Now the coefficient of aqueous conductivity $K = K_0 \theta, K_0 = 0.232$ [9], then the equation (10) becomes,

$$\frac{\partial \theta}{\partial t} = \alpha_0 \frac{\partial}{\partial z}\left(\theta \frac{\partial \theta}{\partial z} \right) + A \frac{\partial \theta}{\partial z}$$

(11)

where; $A = \dfrac{\rho g K_0}{\rho_s}$ = Constant

The equation (11) is non-linear second order partial differential equation, which governs the one-dimensional unsteady flow in unsaturated heterogeneous porous medium in vertically downward direction, over a large basin of geological configuration consider as model.

The appropriate initial and boundary conditions are,

$\theta(0,t) = \theta_0$, for any $t > 0$

(12)

Which is moisture content of water at soil surface $z = 0$.

$\theta(z,0) = \theta_1$, for any $z > 0$

(13)

Which is initial moisture content of water.

$\dfrac{\partial\theta}{\partial z}(0,t) = \omega$, for any $t > 0$

(14)

Which change in moisture content of water is with respected to depth at soil surface $z = 0$. Here $\theta_0 \neq \theta_1$.

4. Approximate Solution of the problem

Further choosing the similarity transformation,

$\theta(z,t) = t.f(\eta); \eta = \dfrac{z}{t}$ [15]

(15)

Using equation (15) in equation (11), the partial differential equation (11) converts into ordinary differential equation together with conditions (12) and (14) are

$f(\eta) - \eta f'(\eta) = \alpha_0 f(\eta) f''(\eta) + \alpha_0 \left(f'(\eta)\right)^2 + Af'(\eta)$

(16)

and

$f(0) = \theta_0/t$, for any $t > 0$

(17)

$f^{'}(0) = \omega$, for any $t > 0$

(18)

The equation (17) and (18) are the sufficient condition to solve the equation (16).

To find successive coefficient's of Maclaurin's series at $\eta = 0$ .Taking nth derivative of equation (16) and solving for $f^{(n+2)}(\eta)$ and evaluating at $\eta = 0$, we have

$$f^{(n+2)}(0) = -\frac{1}{\alpha_0 f(0)}\Big[\alpha_0 f^{n+1}(0) f^{'}(0) + Af^{n+1}(0) + (n-1) f^{n}(0) + \alpha_0 \sum_{k=1}^{n} \binom{n}{k} \left\{ f^{k+1}(0)\, f^{(n-k+1)}(0) + f^{(n-k+2)}(0)\, f^{k}(0) \right\} \Big]$$

(19)

For the solution, it is necessary to determine the derivatives $f^{(n)}(0)$ for all n = 1, 2, 3........ The $f(0)$ from equation (17), $f^{'}(0)$ can be determined by the condition (18) and $f^{''}(0)$ from equation (16). Further all other higher derivatives can be determined from formula (19) by putting n $\geq$1, 2, 3......

Thus the all obtained value of $f(\eta)$ can be computed by Maclaurin's series.

$$f(\eta) = \sum_{k=0}^{\infty} f^{(k)}(0)\, \frac{\eta^k}{k!}$$

(20)

$$f(\eta) = f(0) + \eta f^{'}(0) + \frac{\eta^2}{2!} f^{''}(0) + \frac{\eta^3}{3!} f^{'''}(0) + \frac{\eta^4}{4!} f^{iv}(0) + \text{........}$$

(21)

Now resubstituting value of $f(\eta)$ by using (15), we get

$$\theta(z,t) = f(0) + z\, f^{'}(0) + \frac{z^2}{2t} f^{''}(0) + \frac{z^3}{6t^2} f^{'''}(0) + \frac{z^4}{24t^3} f^{\mathbf{iv}}(0) + \text{....}$$

(22)

Where;

$$f(0) = \theta_0 / t$$

$$f^{'}(0) = \omega$$

$$f^{''}(0) = \frac{1}{\alpha_0 \theta_0}\left[\theta_0 - \alpha_0 \omega^2 t - A\omega t\right]$$

$$f^{'''}(0) = -\frac{t}{(\alpha_0 \theta_0)^2}\left[3\alpha_0 \theta_0 \omega - 3\alpha_0^{\,2} \omega^2 t - 4\alpha_0 A \omega^2 t + A\theta_0 - A^2 \omega t\right]$$

$$f^{iv}(0) = \frac{t}{(\alpha_0 \theta_0)^3}\Big[10\alpha_0^{\,2}\omega^2\theta_0 t - 9\alpha_0^{\,3}\omega^3 t^2 - 12\alpha_0^{\,2} A\omega^3 t^2 + 7\alpha_0 A\omega\theta_0 t - 7\alpha_0 A^2\omega^2 t^2 - 3\alpha_0^{\,2} A\omega^2 t^2$$

$$+A^2\theta_0 t - A^3\omega t^2 - \alpha_0\theta_0^{\,2} - 2\alpha_0(\theta_0 - \alpha_0\omega^2 t - A\omega t)^2)\Big]$$

Equation (22) represents the moisture content of water in vertical groundwater discharge flow through unsaturated heterogeneous porous media with different depth z and time $t > 0$.

5. Numerical and graphical presentation

Numerical and graphical presentations of equation (22) have been obtained by using MATLAB coding. Figure 3 shows the graph of $\theta(z,t)$ vs. z for time $t = 1.0$ to 2.0, and Table 1 represent the numerical data of the moisture content at different depth z for $t \geq 0$.

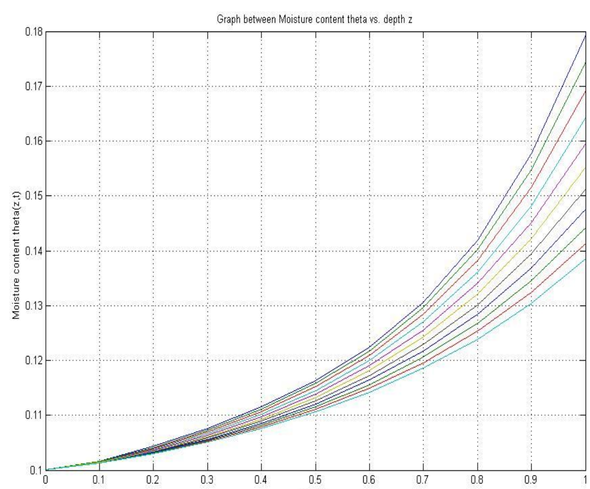

Figure 3: Graph between moisture content at different depth z when t=1.0 to 2.0 fixed

Table 1: Numerical data for the moisture content $\theta(z,t)$ for any depth z and different time t

$\theta(z,t)$											
z/t	**1.0**	**1.1**	**1.2**	**1.3**	**1.4**	**1.5**	**1.6**	**1.7**	**1.8**	**1.9**	**2.0**
0.0	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1
0.1	0.1017	0.1016	0.1015	0.1015	0.1014	0.1014	0.1014	0.1013	0.1013	0.1013	0.1013
0.2	0.1043	0.1041	0.1039	0.1037	0.1036	0.1035	0.1033	0.1032	0.1031	0.1030	0.1030
0.3	0.1076	0.1073	0.1070	0.1067	0.1064	0.1061	0.1059	0.1056	0.1054	0.1052	0.1051
0.4	0.1115	0.1111	0.1107	0.1102	0.1098	0.1093	0.1089	0.1086	0.1082	0.1079	0.1076
0.5	0.1163	0.1158	0.1152	0.1145	0.1139	0.1133	0.1127	0.1121	0.1116	0.1111	0.1106
0.6	0.1224	0.1218	0.1209	0.1200	0.1190	0.1181	0.1172	0.1164	0.1156	0.1149	0.1142
0.7	0.1306	0.1296	0.1284	0.1270	0.1256	0.1242	0.1230	0.1217	0.1206	0.1196	0.1186
0.8	0.1419	0.1403	0.1383	0.1362	0.1341	0.1321	0.1302	0.1284	0.1268	0.1253	0.1239
0.9	0.1576	0.1548	0.1515	0.1483	0.1451	0.1422	0.1394	0.1369	0.1345	0.1324	0.1304
1	0.1793	0.1744	0.1692	0.1642	0.1595	0.1552	0.1512	0.1476	0.1443	0.1413	0.1385

6. Conclusions

The solution (22) of the governing equation (11) together with boundary conditions (12) and (14) represents moisture content of water at any depth z for $t > 0$ when water flow through soil surface $z = 0$ vertically in downward direction in unsaturated heterogeneous porous media over a large basin of geological configuration of model considered. The solution (22) satisfies both conditions (12) and (14). The moisture content of water is expressed as ascending power series of depth z and function of time t with different coefficients $f(0), f'(0), f''(0), f'''(0)$ and $f^{iv}(0)$ which are function of time t. It is finite power series but for our particular interest for study of moisture content of water, we consider first fours term of power series which gives an approximate solution of moisture content of water which is convergence series. The figure of graph (3) shows moisture content at different depth z when $t = 1.0$ to 2.0 fixed. It shows that moisture content $\theta = \theta_0 = 0.1$ initially and then it is increasing when time will increase from 1 to 2 as shown in figure (3) which is physically fact which such flow takes place and its maximum value will be at bottom which given by a thick layer of water table.

This model has great importance in the petroleum engineer concerned with the movement of oil, gas and water through the reservoir of an oil or gas field, to the hydrologist in his study of the migration of underground water, and to the chemical engineer in connection with filtration process, concentration distribution, nuclear waste problem and many other applications.

Nomenclature

θ = Moisture content

ρ_s = The bulk density of the medium

ρ = Density of the fluid

D = Diffusivity

t = Time (s)

z = Depth (m)

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