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PRIME Γ –IDEALS IN DUO Γ-SEMIGROUPS

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ABSTRACT

In this paper the terms left duo Γ- semigroup, right duo Γ- semigroup, duo Γ- semigroup are introduced. It is proved that a Γ-semigroup S is a duo Γ- semigroup if and only if $x\mathbf{\Gamma}S^1 = S^1\mathbf{\Gamma}x$ for all $x \in S$. Further it is proved that every quasi commutative Γ-semigroup is a duo Γ-semigroup. If A is a Γ-ideal in a (Left or right) duo **Γ**-semigroup S, then it is proved that $x, y \in S,\ x\Gamma y \subseteq A \Rightarrow x\Gamma s\Gamma y \subseteq A$. If A is a Γ-ideal in a duo Γ-semigroup S, then it is proved that $A_r(a) = \{ x \in S : a\mathbf{\Gamma}x \subseteq A \}$ is a **Γ**-ideal of S for all $a \in S$ and $A_l(a) = \{ x \in S : x\mathbf{\Gamma}a \subseteq A \}$ is a **Γ**-ideal of S for all $a \in S$. It is proved that if A is a Γ-ideal in a duo **Γ**-semigroup S, then (1) $a\mathbf{\Gamma}b \subseteq A$ if and only if $< a > \mathbf{\Gamma} < b > \subseteq A$ and (2) $a_1\mathbf{\Gamma}a_2\mathbf{\Gamma}.....a_{n-1}\mathbf{\Gamma}a_n \subseteq A$ if and only if $<a_1> \mathbf{\Gamma} <a_2>....\mathbf{\Gamma} < a_n > \subseteq A$. Further it is proved that if A is a Γ-ideal in a duo **Γ**-semigroup S then for any natural number n, $(a\Gamma)^{n-1}a \subseteq A$ implies $(< a > \mathbf{\Gamma})^{n-1} < a > \subseteq A$. If A_1 = the intersection of all completely prime Γ-ideals of S containing A, $A_2 = \{x \in S : (x\Gamma)^{n-1}x \subseteq A$ for some natural number $n\}$, A_3 = the intersection of all prime ideals of S containing A, $A_4 = \{x \in S : (<x>\Gamma)^{n-1} <x> \subseteq A$ for some natural number $n\}$ for a Γ-ideal A of a Γ-semigroup S, then it is proved that A_2 is the minimal completely semiprime Γ-ideal of S containing A, A_4 is the minimal semiprime Γ-ideal of S containing A and $A_1 = A_2 = A_3 = A_4$. It is proved that if A is a **Γ**-ideal of a duo **Γ**-semigroup S, then (1) A is completely prime if and only if A is a prime and (2) A is a completely semiprime if and only if A is a semiprime. If S is a duo **Γ**-semigroup, then it is proved that (1) S is strongly archimedean if and only if archimedean and (2) S is archimedean if and only if S has no proper prime **Γ**-ideals. Further it is proved that if S is a duo **Γ**-semigroup, then the conditions (1) S is strongly Archimedean, (2) S is Archimedean and (3) S has no proper prime **Γ**-ideals are equivalent.

SUBJECT CLASSIFICATION (2010) : 20M07, 20M11, 20M12.

KEY WORDS : left duo Γ-semigroup right duo Γ-semigroup and duo Γ-semigroup.

1. INTRODUCTION:

Γ-semigroup was introduced by Sen and Saha [12] as a generalization of semigroup. KRULL proved that the nil-radical of an ideal A in a commutative ring is equal to the intersection of all minimal prime ideals containing A. SATYANARAYANA has developed some literature on prime ideals and prime radicals for commutative semigroups and obtained KRULL's theorem for commutative semigroups. GIRI and WAZALWAR studied about prime radicals in general semigroups. DHEENA and ELAVARASAN made a study on prime ideals, completely prime ideals, semiprime ideals and completely semiprime ideals in partially ordered Γ-semigroups. MADHUSUDHANA RAO, ANJANEYULU and GANGAHARA RAO [5], [6] and [7] studied about the prime ideals, completely prime ideals, semiprime ideals and completely semiprime ideals, prime radicals in general Γ-semigroups. In this paper we study about the prime ideals, completely prime ideals, semiprime ideals and completely semiprime ideals, prime radicals and generalize the results obtained by SATYANARAYANA in duo Γ-semigroups.

2. PRELIMINARIES:

DEFINITION 2.1 : Let S and Γ be any two non-empty sets. Then S is said to be a **Γ-*semigroup*** if there exist a mapping from $S \times \Gamma \times S$ to S which maps $(a, \gamma, b) \to a\,\gamma\, b$ satisfying the condition : $(a\alpha b)\beta c = a\alpha(b\beta c)$ for all $a, b, c \in$ S and $\alpha, \beta, \gamma \in \Gamma$.

NOTE 2.2 : Let S be a Γ-semigroup. If A and B are two subsets of S, we shall denote the set $\{ a\gamma b : a \in A , b \in B \text{ and } \gamma \in \Gamma \}$ by AΓB.

NOTATION 2.3 : Let S be a Γ -semigroup. If S has an identity, let S^1 = S. If S does not have an identity, let S^1 be the Γ -semigroup S with an identity adjoined, usually denoted by the symbol 1.

DEFINITION 2.4 : A Γ-semigroup S is said to be ***commutative* Γ-semigroup** provided $a\gamma b = b\gamma a$ for all $a,b \in S$ and $\gamma \in \Gamma$.

NOTE 2.5 : If S is a commutative Γ-semigroup then $a\,\Gamma b = b\,\Gamma a$ for all $a, b \in S$.

NOTE 2.6 : Let S be a Γ-semigroup and $a, b \in$ S and $\alpha \in \Gamma$. Then $a\alpha a\alpha b$ is denoted by $(a\alpha)^2 b$ and consequently $a\,\alpha\, a\,\alpha\, a\,\alpha$…..(*n terms*)$b$ is denoted by $(a\alpha)^n b$.

DEFINITION 2.7: A Γ-semigroup S is said to be ***quasi commutative*** provided for each $a,b \in$ S, there exists a natural number *n* such that $a\gamma b = (b\gamma)^n a \;\; \forall \gamma \in \Gamma$.

NOTE 2.8 : If a Γ-semigroup S is ***quasi commutative*** then for each $a,b \in$ S , there exists a natural number *n* such that, $a\Gamma b = (b\,\Gamma)^n a$.

THEOREM 2.9 : If S is a commutative Γ-semigroup then S is a quasi commutative Γ-semigroup.

DEFINITION 2.10 : A Γ-semigroup S is said to be ***normal*** provided $a\alpha S = S\alpha a \;\; \forall \alpha \in \Gamma$ and $\forall a \in S$.

NOTE 2.11 : If a Γ-semigroup S is *normal* then $a\Gamma S = S\Gamma a$ for all $a \in$ S.

THEOREM 2.12: If S is a quasi commutative Γ-semigroup then S is a normal Γ-semigroup.

COROLLARY 2.13 : Every commutative Γ-semigroup is a normal Γ-semigroup.

DEFINITION 2.14 : A nonempty subset A of a Γ-semigroup S is said to be a ***left* Γ-*ideal*** of S if $s \in S, a \in A, \alpha \in \Gamma$ implies $s\alpha a \in A$.

NOTE 2.15 : A nonempty subset A of a Γ-semigroup S is a left Γ- ideal of S iff SΓA⊆A.

THEOREM 2.16 : The nonempty intersection of any two left Γ-ideals of a Γ-semigroup S is a left Γ-ideal of S.

THEOREM 2.17: The nonempty intersection of any family of left Γ-ideals of a Γ-semigroup S is a left Γ-ideal of S.

THEOREM 2.18: The union of any two left Γ-ideals of a Γ-semigroup S is a left Γ-ideal of S.

THEOREM 2.19 : The union of any family of left Γ-ideals of a Γ-semigroup S is a left Γ-ideal of S.

DEFINITION 2.20 : A nonempty subset A of a Γ-semigroup S is said to be a ***right* Γ-*ideal*** of S if $s \in S, a \in A, \alpha \in \Gamma$ implies $a\alpha s \in A$.

NOTE 2.21 : A nonempty subset A of a Γ-semigroup S is a right Γ- ideal of S iff AΓS⊆A.

THEOREM 2.22 : The nonempty intersection of any two right Γ-ideals of a Γ-semigroup S is a right Γ-ideal of S.

THEOREM 2.23 : The nonempty intersection of any family of right Γ-ideals of a Γ-semigroup S is a right Γ-ideal of S.

THEOREM 2.24 : The union of any two right Γ-ideals of a Γ-semigroup S is a right Γ-ideal of S.

THEOREM 2.25 : The union of any family of right Γ-ideals of a Γ-semigroup S is a right Γ-ideal of S.

DEFINITION 2.26 : A nonempty subset A of a Γ-semigroup S is said to be a ***two sided* Γ- *ideal*** or simply a **Γ- *ideal*** of S if $s \in S$, $a \in A$, $\alpha \in \Gamma$ imply $s\alpha a \in A$, $a\alpha s \in A$.

NOTE 2.27 : A nonempty subset A of a Γ-semigroup S is a two sided Γ-ideal iff it is both a left Γ-ideal and a right Γ- ideal of S.

THEOREM 2.28 : The nonempty intersection of any two Γ-ideals of a Γ-semigroup S is a Γ-ideal of S.

THEOREM 2.29 : The nonempty intersection of any family of Γ-ideals of a Γ-semigroup S is a Γ-ideal of S.

THEOREM 2.30 : The union of any two Γ-ideals of a Γ-semigroup S is a Γ-ideal of S.

THEOREM 2.31 : The union of any family of Γ-ideals of a Γ-semigroup S is a Γ-ideal of S.

DEFINITION 2.32 : A Γ-ideal A of a Γ-semigroup S is said to be a ***proper* Γ-*ideal*** of S if A is different from S.

DEFINITION 2.33 : A Γ-ideal A of a Γ-semigroup S is said to be a ***trivial* Γ-*ideal*** provided S\A is singleton.

DEFINITION 2.34 : A Γ-ideal A of a Γ-semigroup S is said to be a ***maximal* Γ-*ideal*** provided A is a proper Γ-ideal of S and is not properly contained in any proper Γ-ideal of S.

THEOREM 2.35 : If S is a Γ-semigroup with unity 1 then the union of all proper Γ-ideals of S is the unique maximal Γ-ideal of S.

DEFINITION 2.36 : Let S be a Γ-semigroup and A be a nonempty subset of S. The smallest Γ-ideal of S containing A is called ***Γ-ideal of* S *generated by A***.

THEOREM 2.37 : The Γ-ideal of a Γ-semigroup S generated by a nonempty subset A is the intersection of all Γ-ideals of S containing A.

DEFINITION 2.38 : A Γ- ideal A of a Γ-semigroup S is said to be a ***principal* Γ- *ideal*** provided A is a Γ- ideal generated by $\{a\}$ for some $a \in$ S. It is denoted by J[a] or $\langle a\rangle$.

THEOREM 2.39 : If S is a Γ-semigroup and $a \in$ S then $\langle a \rangle = a \cup a\Gamma S \cup S\Gamma a \cup S\Gamma a\Gamma S$.

NOTE 2.40 : If S is a Γ-semigroup and $a \in S$, then $\langle a\rangle = a \cup a\Gamma S \cup S\Gamma a \cup S\Gamma a\Gamma S = S^1\Gamma a\, \Gamma S^1$.

DEFINITION 2.41 : A Γ- ideal P of a Γ-semigroup S is said to be a ***completely prime* Γ- *ideal*** provided $x, y \in$ S and $x\Gamma y \subseteq$ P implies either $x \in$ P or $y \in$ P.

DEFINITION 2.42 : A Γ- ideal P of a Γ-semigroup S is said to be a ***prime* Γ- *ideal*** provided A, B are two Γ-ideals of S and $A\Gamma B \subseteq P \Rightarrow$ either $A \subseteq P$ or $B \subseteq P$.

THEOREM 2.43 : If P is a Γ- ideal of a Γ-semigroup S, then the following conditions are equivalent.

(1) If A, B are Γ- ideals of S and $A\Gamma B \subseteq P$ then either $A \subseteq P$ or $B \subseteq P$.
(2) If $a, b \in$ S such that $a\Gamma S^1\Gamma b \subseteq$ P, then either $a \in$ P or $b \in$ P.

COROLLARY 2.44 : A Γ- ideal P of a Γ-semigroup S is a prime Γ- ideal iff $a, b \in$ S such that $a\Gamma S^1\Gamma b \subseteq$ P, then either $a \in$ P or $b \in$ P.

THEOREM 2.45 : Every completely prime Γ-ideal of a Γ-semigroup S is a prime Γ-ideal of S.

THEOREM 2.46 : Let S be a commutative Γ-semigroup. A Γ-ideal P of S is prime Γ-ideal if and only if P is a completely prime Γ-ideal.

DEFINITION 2.47: A Γ-ideal A of a Γ-semigroup S is said to be a ***completely semiprime* Γ- *ideal*** provided $x\Gamma x \subseteq A$; $x \in S$ implies $x \in A$.

THEOREM 2.48 : Every completely prime Γ-ideal of a Γ-semigroup S is a completely semiprime Γ-ideal of S.

THEOREM 2.49 : The nonempty intersection of any family completely prime Γ-ideals of a Γ-semigroup S is a completely semiprime Γ-ideal of S.

DEFINITION 2.50 : A Γ- ideal A of a Γ-semigroup S is said to be a ***semiprime* Γ- *ideal*** provided $x \in S$, $x\Gamma S^1\Gamma x \subseteq A$ implies $x \in A$.

THEOREM 2.51 : Every completely semiprime Γ-ideal of a Γ-semigroup S is a semiprime Γ-ideal of S.

THEOREM 2.52 : Let S be a commutative Γ-semigroup. A Γ-ideal A of S is completely semiprime iff semiprime.

THEOREM 2.53 : Every prime Γ-ideal of a Γ-semigroup S is a semiprime Γ-ideal of S.

THEOREM 2.54 : The nonempty intersection of any family of prime Γ-ideals of a Γ-semigroup S is a semiprime Γ-ideal of S.

NOTATION 2.55 : If A is a Γ-ideal of a Γ-semigroup S, then we associate the following four types of sets.

A_1 = The intersection of all completely prime Γ-ideals of S containing A.

$A_2 = \{x \in S : (x\Gamma)^{n-1}x \subseteq A$ for some natural number $n\}$

A_3 = The intersection of all prime ideals of S containing A.

$A_4 = \{x \in S : (< x >\Gamma)^{n-1} < x > \subseteq A$ for some natural number $n\}$

THEOREM 2.56 : If A is a Γ- ideal of a Γ-semigroup S, then $A \subseteq A_4 \subseteq A_3 \subseteq A_2 \subseteq A_1$.

THEOREM 2.57 : If A is a Γ-ideal of a commutative Γ-semigroup S, then $A_1 = A_2 = A_3 = A_4$.

NOTE 2.58 : If A is a Γ-ideal in a noncommutative Γ-semigroup, then A_1, A_2, A_3, A_4 need not be equal.

EXAMPLE 2.59 : Let S be the free Γ-semigroup generated by two alphabets a, b. It is clear that $A = S\Gamma a\Gamma a\Gamma S$ is a Γ-ideal in S. Since $(a\Gamma)^3a \subseteq S\Gamma a\Gamma a\Gamma S = A$, We have $a \in A_2$. Evidently $(a\Gamma b\Gamma)^{n-1}a\Gamma b \not\subseteq S\Gamma a\Gamma a\Gamma S$ for all natural number n and thus $a\Gamma b \not\subseteq A_2$. Thus A_2 is not a Γ-ideal in S. Therefore $A_1 \neq A_2$ and $A_2 \neq A_3$.

DEFINITION 2.60 : If A is a Γ-ideal of a Γ-semigroup S, then the intersection of all prime Γ-ideals of S containing A is called ***prime* Γ*-radical*** or simply **Γ*-radical*** of A and it is denoted by $\sqrt{\mathbf{A}}$ or ***rad A***.

DEFINITION 2.61 : If A is a Γ-ideal of a Γ-semigroup S, then the intersection of all completely prime Γ-ideals of S containing A is called ***complete prime* Γ*-radical*** or simply ***complete* Γ*-radical*** of A and it is denoted by ***c. rad A***.

NOTE 2.62 : If A is a Γ-ideal of a Γ-semigroup S then ***rad*** $\mathbf{A = A_3}$ and ***c.rad*** $\mathbf{A = A_4}$**.**

THEOREM 2.63 : If A is a Γ-ideal of a commutative Γ-semigroup S, then *rad* A = *c.rad* A

THEOREM 2.64 : If A and B are any two Γ-ideals of a Γ-semigroup S, then

(i) $A \subseteq B \Rightarrow \sqrt{(A)} \subseteq \sqrt{(B)}$.

(ii) $\sqrt{(A\Gamma B)} = \sqrt{(A \cap B)} = \sqrt{(A)} \cap \sqrt{(B)}$.

(iii) $\sqrt{(\sqrt{(A)})} = \sqrt{(A)}$.

THEOREM 2.65 : If A and B are any two Γ-ideals of a Γ-semigroup S, then

(i) A ⊆ B ⇒ *c.rad* A ⊆ *c.rad* B.

(ii) *c.rad* (AΓB) = *c.rad* (A∩B) = *c.rad* (A) ∩ *c.rad* (B).

(iii) *c.rad* (*c.rad* A) = *c.rad* A.

THEOREM 2.66 : If A is a Γ-ideal of a Γ-semigroup S then *c.rad* A is a completely semiprime Γ-ideal of S.

THEOREM 2.67 : If A is a Γ-ideal of a Γ-semigroup S then $\sqrt{A}$ is a semiprime Γ-ideal of S.

THEOREM 2.68 : A Γ- ideal Q of Γ-semigroup S is a semiprime Γ- ideal of S iff $\sqrt{(Q)} = Q$.

DEFINITION 2.69 : A Γ- semigroup S is said to be an ***archimedian* Γ*- semigroup*** provided for any $a,b \in S$, there exists a natural number n such that $(a\Gamma)^{n-1}a \subseteq \langle b \rangle$.

DEFINITION 2.70 : A Γ-semigroup S is said to be a ***strongly archimedean* Γ*-semigroup*** provided for any a, $b \in S$, there is a natural number n such that $(\langle a \rangle\Gamma)^{n-1} \langle a \rangle \subseteq \langle b \rangle$.

THEOREM 2.71 : Every strongly archimedean Γ-semigroup is an archimedean Γ-semigroup.

3. DUO Γ- SEMIGROUPS:

DEFINITION 3.1 : A Γ- semigroup S is said to be a ***left duo* Γ*- semigroup*** provided every left Γ- ideal of S is a two sided Γ- ideal of S.

DEFINITION 3.2 : A Γ- semigroup S is said to be a ***right duo* Γ*- semigroup*** provided every right Γ- ideal of S is a two sided Γ- ideal of S.

DEFINITION 3.3 : A Γ- semigroup S is said to be a ***duo* Γ- *semigroup*** provided it is both a left duo Γ- semigroup and a right duo Γ- semigroup.

THEOREM 3.4 : A Γ-semigroup S is a duo Γ- semigroup if and only if $x\Gamma S^1 = S^1\Gamma x$ for all $x \in S$.

Proof: Suppose that S is a duo Γ-Semigroup and $x \in S$.
Let $t \in x\Gamma S^1$. Then $t=x\gamma s$ for some $s \in S^1$, $\gamma \in \Gamma$.

Since $S^1\Gamma x$ is a left Γ-ideal of S, $S^1\Gamma x$ is a Γ-ideal of S.

So $x \in S^1\Gamma x,\ \gamma \in \Gamma,\ s \in S,\ S^1\Gamma x$ is a Γ-ideal $\Rightarrow x\gamma s \in S^1\Gamma x \Rightarrow t \in S^1\Gamma x$.

Therefore $x\Gamma S^1 \subseteq S^1\Gamma x$. Similarly we can prove that $S^1\Gamma x \subseteq x\Gamma S^1$ .Therefore $S^1\Gamma x = x\Gamma S^1$.

Conversely suppose that $S^1\Gamma x = x\Gamma S^1$ for all $x \in S$. Let A be a left Γ-ideal of S.

Let $x \in A$, $s \in S$ and $\alpha \in \Gamma$. Then $x\alpha s \in x\Gamma S^1 = S^1\Gamma x \Rightarrow x\alpha s = t\beta x$ for some $t \in S^1$, $\beta \in \Gamma$.

$x \in A,\ t \in S,\ \beta \in \Gamma$, A is a left Γ-ideal of S $\Rightarrow t\beta x \in A \Rightarrow x\alpha s \in A$.

Therefore A is a right Γ-ideal of S and hence A is a Γ-ideal of S.

Therefore S is left duo Γ-semigroup.

THEOREM 3.5 : Every quasi commutative Γ-semigroup is a duo Γ-semigroup.

Proof : Suppose that S is a quasi commutative Γ-semigroup. Then for $a, b \in S$, thereexists $n \in N$ such that $a\gamma b = (b\gamma)^n a$ for all $\gamma \in \Gamma$. Let A be a left Γ-ideal of S. Therefore $S\Gamma A \subseteq A$. Let $a \in A$ and $s \in S$. Since S is a quasi commutative Γ-semigroup, there exists a natural number n such that $a\Gamma s = (s\Gamma)^n a \subseteq S\Gamma A \subseteq A$. Therefore $a\Gamma s \subseteq A$. for all $a \in A$ and $s \in S$ and hence $A\Gamma S \subseteq A$. Thus A is right Γ-ideal of S. Therefore S is a left duo Γ-semigroup. Similarily we can prove that S is a right duo Γ-semigroup. Therefore every quasi commutative Γ-semigroup is a duo Γ-semigroup.

THEOREM 3.6 : Every normal Γ- semigroup is a duo Γ- semigroup.

Proof: Suppose that S is normal Γ-semigroup.
Then $a\Gamma S = S\Gamma a$ for all $a \in S \Rightarrow a\Gamma S^1 = S^1\Gamma a$ for all $a \in S$.

By theorem 3.4, S is a duo Γ-semigroup.

THEOREM 3.7 : Every commutative Γ-semigroup is a duo Γ-semigroup.

Proof : Suppose that S is a commutative Γ-semigroup. By corollary 2.13, S is a normal Γ-semigroup. By theorem 3.6, S is a duo Γ-semigroup.

THEOREM 3.8: If A is a Γ-ideal in a right duo Γ-semigroup S, then $A_r(a) = \{ x \in S : a\Gamma x \subseteq A \}$ is a Γ-ideal of S for all $a \in S$.

Proof : Let $x \in A_r(a)$ and $s \in S$. $x \in A_r(a) \Rightarrow a\Gamma x \subseteq A$.
$a\Gamma x \subseteq A$, $s \in S$, A is a Γ-ideal $\Rightarrow a\Gamma x\Gamma s \subseteq A \Rightarrow x\Gamma s \subseteq A_r(a)$.

Therefore $A_r(a)$ is a right Γ-ideal of S.

Since S is a right duo Γ-semigroup, $A_r(a)$ is a Γ-ideal of S.

THEOREM 3.9: If A is a Γ-ideal in a left duo Γ-semigroup S, then $A_l(a) = \{ x \in S : x\Gamma a \subseteq A \}$ is a Γ-ideal of S for all $a \in S$.

Proof : Let $x \in A_l(a)$ and $s \in S$. $x \in A_l(a) \Rightarrow x\Gamma a \subseteq A$.
$x\Gamma a \subseteq A$, $s \in S$, A is a Γ-ideal $\Rightarrow s\Gamma x\Gamma a \subseteq A \Rightarrow s\Gamma x \subseteq A_l(a)$.

Therefore $A_l(a)$ is a left Γ-ideal of S. Since S is a left duo Γ-semigroup, $A_l(a)$ is a Γ-ideal of S.

COROLLARY 3.10 : If A is a Γ-ideal in a duo Γ-semigroup S, then $A_l(a) = \{ x \in S : x\Gamma a \subseteq A \}$ and $A_r(a) = \{ x \in S : a\Gamma x \subseteq A \}$ are Γ-ideals of S for all $a \in S$.

THEOREM 3.11 : If A is a Γ-ideal in a left duo Γ-semigroup S and $x, y \in S$, then $x\Gamma y \subseteq A$ implies $x\Gamma s\Gamma y \subseteq A$ for all $s \in S$.

Proof : Suppose that $x\Gamma y \subseteq A$. Let $s \in S$.
$x\Gamma y \subseteq A \Rightarrow x \in A_l(y)$.

$x \in A_l(y)$, $s \in S$, $A_l(y)$ is a Γ-ideal of S $\Rightarrow x\Gamma s \subseteq A_l(y) \Rightarrow x\Gamma s\Gamma y \subseteq A$.

THEOREM 3.12 : If A is a Γ-ideal in a right duo Γ-semigroup S and $x, y \in S$, then $x\Gamma y \subseteq A$ implies $x\Gamma s\Gamma y \subseteq A$.

Proof : Suppose that $x\Gamma y \subseteq A$. Let $s \in S$.
$x\Gamma y \subseteq A \Rightarrow y \in A_r(x)$.

$y \in A_r(x)$, $s \in S$, $A_r(x)$ is a Γ-ideal of S $\Rightarrow s\Gamma y \subseteq A_r(x) \Rightarrow x\Gamma s\Gamma y \subseteq A$.

COROLLARY 3.13 : If A is a Γ-ideal in a duo Γ-semigroup S and $x, y \in S$, then $x\Gamma y \subseteq A$ implies $x\Gamma s\Gamma y \subseteq A$.

THEOREM 3.14 : Let A be a Γ-ideal in a duo Γ-semigroup S and $a, b \in S$. Then $a\Gamma b \in A$ if and only if $< a > \Gamma < b > \subseteq A$.

Proof : Suppose that $< a > \Gamma < b > \subseteq A$.
Then $a\Gamma b \subseteq < a > \Gamma < b > \subseteq A$.

Conversely suppose that $a\Gamma b \subseteq A$.

Since S is a duo Γ-semigroup. By corollary 3.13, $a\Gamma b \subseteq A \Rightarrow a\Gamma s\Gamma b \subseteq A$ for all $s \in S$ $\Rightarrow a\Gamma S^1\Gamma b \subseteq A$. Since A is a Γ-ideal, $a\Gamma S^1\Gamma b \subseteq A \Rightarrow S^1\Gamma a\, \Gamma S^1\, \Gamma\, b\, \Gamma S^1 \subseteq A \Rightarrow\ < a > \Gamma < b > \subseteq A$.

THEOREM 3.15 : Let A be a Γ-ideal in a duo Γ-semigroup S. Then $a_1\Gamma a_2\Gamma \ldots a_{n-1}\Gamma a_n \subseteq A$ if and only if $< a_1 > \Gamma < a_2 > \ldots \Gamma < a_n > \subseteq A$.

Proof : Suppose that $< a_1 >\Gamma< a_2 > \ldots \Gamma< a_n > \subseteq A$.
Then $a_1 \Gamma a_2 \Gamma \ldots a_{n-1} \Gamma a_n \subseteq < a_1 > \Gamma < a_2 > \ldots \Gamma < a_n > \subseteq A$.

Conversely suppose that $a_1 \Gamma a_2 \Gamma \ldots a_{n-1} \Gamma a_n \subseteq A$.

Then for any $t \in < a_1 > \Gamma < a_2 > \ldots \Gamma < a_n >$, we have

$t = s_1\alpha_1 a_1\beta_1 s_2\alpha_2 a_2\beta_2 \ldots \alpha_n a_n\beta_n s_{n+1}$, where $s_i \in S$ and $\alpha_i, \beta_i \in \Gamma$.

Since $x, y \in S$, $x\Gamma y \subseteq A \Rightarrow x\Gamma s\Gamma y \subseteq A$, we have $t \in A$.

Therefore $< a_1 > \Gamma < a_2 > \ldots \Gamma < a_n > \subseteq A$.

COROLLARY 3.16 : Let A be a Γ-ideal in a duo Γ-semigroup S. Then for any natural number n, $(a\Gamma)^{n-1}a \subseteq A$ implies $(< a > \Gamma)^{n-1} < a > \subseteq A$.

Proof : The proof follows from theorem 3.18, by taking $a_1 = a_2 = a_3 = \ldots = a_n = a$.

THEOREM 3.17 : If A is a Γ-ideal in a duo Γ-semigroup S, then $A_4 = \{x \in S : (< x >\Gamma)^{n-1} < x > \subseteq A$ for some $n \in N\}$ is the minimal semiprime Γ-ideal of S containing A.

Proof : Clearly $A \subseteq A_4$ and hence A_4 is nonempty subset of S. Let $x \in A_4$ and $s \in S$.
Since $x \in A_4$, $(< x >\Gamma)^{n-1} < x > \subseteq A$ for some $n \in N$.
Now $(< x\Gamma s >\Gamma)^{n-1} < x\Gamma s > \subseteq (< x >\Gamma)^{n-1} < x > \subseteq A$ and
$(s\Gamma x >\Gamma)^{n-1} < s\Gamma x > \subseteq (< x >\Gamma)^{n-1} < x > \subseteq A$ implies $x\Gamma s, s\Gamma x \subseteq A_4$.
Therefore A_4 is a Γ-ideal of S containing A. Let $x \in S$ such that $(< x > \Gamma) < x > \subseteq A_4$.
Then $(< x >\Gamma< x > \Gamma)^{n-1} < x > \Gamma< x > \subseteq A$ implies $(< x > \Gamma)^{2n-1} < x > \subseteq A \Rightarrow x \in A_4$.
Thus A_4 is semiprime Γ-ideal of S containing A.
Let Q be a semiprime Γ-ideal of S containing A. Let $x \in A_4$. Then $(< x >\Gamma)^{n-1} < x > \subseteq A$ for some $n \in N$. Since $A \subseteq Q$, then $(< x >\Gamma)^{n-1} < x > \subseteq Q$ for some $n \in N$.

Since Q is a semiprime Γ-ideal of S, $(< x >\Gamma)^{n-1} < x > \subseteq Q \Rightarrow x \in Q$.

Therefore $A_4 \subseteq Q$ and hence A_4 is the minimal semiprime Γ-ideal of S containing A.

THEOREM 3.18 : If A is a Γ-ideal in a duo Γ-semigroup S. Then $A_2 = \{x \in S : (x\Gamma)^{n-1} x \subseteq A$ for some $n \in N\}$ is the minimal completely semiprime Γ-ideal of S containing A.

Proof : Clearly $A \subseteq A_2$ and hence A_2 is nonempty subset of S. Let $x \in A_2$ and $s \in S$.
Since $x \in A_2$, $(x\Gamma)^{n-1} x \subseteq A$ for some $n \in N$.
Now $(x\Gamma s)^{n-1} x\Gamma s \subseteq A$ and
$(s\Gamma x)^{n-1} s\Gamma x \subseteq A$ implies $x\Gamma s, s\Gamma x \in A_2$.
Therefore A_2 is a Γ-ideal of S containing A. Let $x \in S$ such that $\Gamma x \subseteq A_2$.
Then $(x\Gamma x \Gamma)^{n-1} x \Gamma x \subseteq A$ implies $(x\Gamma)^{2n-1}x \subseteq A \Rightarrow x \in A_2$.
Thus A_2 is a completely semiprime Γ-ideal of S containing A. Let P be a completely semi prime

Γ-ideal of S containing A. Let $x \in A_2$. Then $(x\Gamma)^{n-1} x \subseteq A$ for some $n \in N$. Since $A \subseteq P$, then $(x\Gamma)^{n-1} x \subseteq P$, for some $n \in N$.
Since P is completely semiprime Γ-ideal of S, $(x\Gamma)^{n-1} x \subseteq P \Rightarrow x \in P$.
Therefore $A_2 \subseteq P$ and hence A_2 is the minimal completely semiprime Γ-ideal of S containing A.

THEOREM 3.19 : If A is a Γ-ideal in a duo Γ-semigroup S then $A_2 = A_4$.

Proof : By theorem 2.56, we have $A_4 \subseteq A_2$. Let $x \in A_2$. Then $(x\Gamma)^{n-1} x \subseteq A$ for some $n \in N$. By corollary 3.16, $(< x > \Gamma)^{n-1} < x > \subseteq A$ and hence $x \in A_4$. Therefore $A_2 \subseteq A_4$ and hence $A_2 = A_4$.

THEOREM 3.20 : A Γ-ideal A of a duo Γ-semigroup S is completely prime Γ-ideal if and only if A is a prime Γ-ideal.

Proof : Suppose that A is a completely prime **Γ**-ideal of a duo **Γ**-semigroup S.
By theorem 2.45, A is a prime **Γ**-ideal of S.

Conversely suppose that A is a prime **Γ**-ideal.

Let $x, y \in S$ and $x\Gamma y \subseteq A$.Then by theorem3.13, $x \Gamma S^1 \Gamma y \subseteq A$.

Since A is a prime Γ-ideal of S, $x \Gamma S^1 \Gamma y \subseteq A \Rightarrow x \in A$ or $y \in A$.

Therefore A is a completely prime **Γ**-ideal of S.

THEOREM 3.21 : If A is a Γ-ideal in a duo Γ-semigroup S then $A_1 = A_2 = A_3 = A_4$.

Proof : By theorem 2.56, we have $A \subseteq A_4 \subseteq A_3 \subseteq A_2 \subseteq A_1$. By theorem 3.20, in a duo **Γ**-semigroup S, a **Γ**-ideal A is a prime **Γ**-ideal iff A is completely prime **Γ**-ideal. So $A_1 = A_3$. By theorem 3.19, $A_2 = A_4$. Therefore $A_1 = A_2 = A_3 = A_4$.

THEOREM 3.22 : A Γ-ideal A of a duo Γ-semigroup S is a completely semiprime Γ-ideal of S if and only if A is a semiprime Γ-ideal.

Proof: Suppose that A is a completely semiprime **Γ**-ideal of a duo **Γ**-semigroup S.
By theorem 2.51, A is a semiprime **Γ**-ideal of S.

Conversely suppose that A is a semiprime **Γ**-ideal .

Let $x\Gamma x \subseteq A$ and $x \in S$.Then by theorem 3.13, $x\Gamma S^1 \Gamma x \subseteq A$.

Since A is a semiprime Γ-ideal of S, $x\Gamma S^1 \Gamma x \subseteq A \Rightarrow x \in A$.

Therefore A is a completely semiprime Γ-ideal of S.

THEOREM 3.23 : If S is a duo Γ-semigroup, then S is strongly archimedean if and only if archimedean.

Proof : Suppose that S is strongly Archimedean.

Then for any $a,b \in S$, there is a natural number n such that $(< a > \Gamma)^{n-1} < a > \subseteq < b >$.
Therefore $(a\Gamma)^{n-1} a \subseteq (< a > \Gamma)^{n-1} < a > \subseteq < b >$ and hence S is Archimedean.

Conversely suppose that S is archemedian. Let $a,b \in$ S. Since S is archemedian, there exists a natural number n such that $(< a >\Gamma)^{n-1} < a > \subseteq\ < b > \subseteq S\,\Gamma b\,\Gamma S$. Since $S\,\Gamma b\,\Gamma S$ is a Γ-ideal of a duo Γ-semigroup S, by corollary 3.16, $(a\Gamma)^{n-1}a \subseteq S\,\Gamma b\,\Gamma S \Rightarrow (< a >\Gamma)^{n-1} < a > \subseteq S\,\Gamma b\,\Gamma S$. Therefore S is a strongly Archimedean duo **Γ**-semigroup**.**

THEOREM 3.24 : If S is a duo Γ-semigroup, then S is archimedean if and only if S has no proper prime Γ-ideals.

Proof : Suppose that S is archimedean Γ-semigroup. Let P be prime Γ-ideal of S. Let $a,b \in$ S. Since P is Γ-ideal, $S\Gamma a\Gamma S \subseteq P$. Since S is archemedian, $(b\Gamma)^{n-1} \subseteq S\Gamma a\Gamma S$ for some natural number n. Thus $(b\Gamma)^{n-1} \subseteq S\Gamma a\Gamma S \subseteq P$. Since S is a duo Γ-semigroup, by theorem 3.24, P is completely prime. Thus $(b\Gamma)^{n-1}b \subseteq P \Longrightarrow b \in$ P. Hence $S = P$. Therefore S has no proper prime Γ-ideals. Conversely suppose that S has no proper prime Γ-ideals. Then for any $b \in$ S, the intersection of all prime Γ-ideals of S containing $B = < b >$ is S itself. Therefore $B_3 = S$. We have $B_4 = \{x \in S : (< x > \Gamma)^{n-1} < x > \subseteq < b > \text{ for some } n \in N\} = S$. Therefore for any $a \in$ S, $(< a >\Gamma)^{n-1} < a > \subseteq < b >$ for some natural number n. So $(< a >\Gamma)^{n-1} < a > \subseteq S\Gamma b\Gamma S$. Thus S is strongly archimedian. Hence by theorem 3.23, S is archimedian.

COROLLARY 3.25 : If S is a duo Γ-semigroup, then the conditions (1) S is strongly Archimedean, (2) S is Archimedean and (3) S has no proper prime Γ-ideals are equivalent.

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