Available online at www.internationaleJournals.com

International eJournals

INTERNATIONAL eJOURNAL OF MATHEMATICS AND ENGINEERING

www.InternationaleJournals.com

International eJournal of Mathematics and Engineering
179 (2012) 1705 – 1712

Transient Inverse Heat Conduction Problem

T. Ramakrishna Goud[1] and D.Ramamurthy[2]
[1]Dept. of Mathematics, Osmania Unuiversity, Hyderabad. A.P. India.
Email: tamtamrkrishna@gmail.com
[2](Retd) Professor Dept. of Mathematics, Osmania Unuiversity, Hyderabad, A.P. India.
Email: drmurthy2k1@yahoo.co.in

Abstract:

In inverse heat conduction problem we estimate the surface temperature or heat flux history given one or more measured temperature histories inside a heat conducting body, here we have considered the problem of determining the temperature and heat flux at the surface of a solid when the temperature at an interior location is a prescribed function of interior location is a prescribed function of time. The theory is able to accommodate an initial temperature distribution which varies arbitrarily with position throughout the solid. The accuracy of the method is demonstrated by a numerical example.

Keywords: Inverse Heat Conduction, Heat Flux, Dynamic Heat Transfer, Efferically Symmetric, Thermal Diffusivity.

AMS Mathematics subject classification: 34B07, 34B40

1. Introduction

If the heat flux or temperature histories at the surface of a solid are known as functions of time, then the temperature distribution can be found. This is called Direct Heat Conduction problem. In many dynamic heat transfer situations, the surface heat flux and temperature histories of a solid must be determined from transient temperature measurements at one or more interior locations. This is called an Inverse Heat Conduction problem and is much more difficult to solve analytically than the direct problem. In inverse heat conduction problem, we estimate the surface heat flux history given one or more measured temperature histories inside a heat conducting body.

The Space research had given considerable impetus to the study of the inverse heat Conduction problems. Another important research area that extensively requires solutions of Inverse Heat Conduction problems is the testing of Nuclear Reactor components. The concept of Inverse Heat Conduction is useful in the areas of periodic heating in combustion chambers of internal combustion engines, solidification of glass, indirect calarimetry for laboratory use, Transient boiling curve studies etc. Stolz[7], Beck[1], Blackwell[2], Imber[4], Mulholland[6], Williams and Curry[8] and others have done considerable work in Inverse Heat Conduction. Estimation of the heating history experienced by a Missile reentering the Earth's atmosphere from space is another important Inverse Heat Conduction problem.

There are many approaches to the Inverse Heat Conduction problems. Integral Transform technique can be used to solve linear problems. Finite differences, and Finite element methods can be used due to their ability to treat non-linear problems. Burggraf[3], Langford[5] and others have obtained exact solutions to some important Inverse Heat Conduction problems. Here we discuss some Inverse Heat Transfer Models.

2. Transient Inverse Heat Conduction

Here we have considered the problem of determining the temperature and heat flux at the surface of a solid when the temperature at an interior location is a prescribed function of interior location is a prescribed function of time. The theory is able to accommodate an initial temperature distribution which varies arbitrarily with position throughout the solid. The accuracy of the method is demonstrated by a numerical example.

Stolz [7] had given an approach to find the temperature solution at an interior point corresponding to a sequence of step changes in heat flux, the first step being applied at $t = 0$, the second step at $t = \Delta t$, the third at $t = 2\Delta t$ and so on. With such expression for the temperature, working backwards, utilizing a prescribed temperature history at an interior point as input he solved for the successive steps in surface heat flux. In this approach, the numerical computation is tedious because it is necessary to know the value of an infinite series at every stage of the computation. Stolz method is restricted to situations in which the initial temperature of the body is uniform. Sparrow E.M. and others have given different approach to the Inverse problems of transient heat conduction. This method is more general one and computationally simpler. Here we can consider an initial non uniform temperature distribution in the body also.

3. Spherically Symmetric Transient Heat Conduction

We want to determine surface temperature concerning to spherically symmetric transient heat conduction. The energy conservation equation appropriate to spherically symmetric transient heat conduction is

$$\frac{\partial^2(\rho T)}{\partial \rho^2} = \frac{\partial(\rho T)}{\partial \tau} \qquad (1)$$

Where ρ and τ, respectively, represent a dimensionless radius and dimensionless time, given by

$$\rho = \frac{r}{R}$$
$$\tau = \frac{\alpha t}{R^2} \tag{2}$$

Here R is the radius of the sphere and α is thermal diffusivity.

First we consider the case in which the initial temperature T_i is uniform. Generalisation to include T_i varying with r will also be considered later. Applying Laplace transform to equation (1) and using uniform initial temperature T_i and using regularity boundary condition at r = 0, we get.

$$\bar{\theta}(\rho, S) = \frac{A}{\rho} Sinh(\sqrt{S}.\rho) \tag{3}$$

Here S is the Laplace transform parameter and $\bar{\theta}(\rho, S)$ is the Laplace transform of θ = T – T_i. In this Inverse Heat Conduction problem, we take the temperature at an interior position $\overset{*}{\rho} < 1$ is prescribed as a function of time.

i.e. $$\theta(\rho^*, \tau) = f(\tau)$$

$$\therefore \bar{\theta}(\rho^*, S) = \bar{f}(S) \tag{4}$$

From equations (3) and (4), we get

$$A = \rho^* \bar{f}(S) / Sinh\sqrt{S}\rho^* \tag{5}$$

$$\therefore \bar{\theta}(\rho, S) = \bar{f}(S)\rho^* Sinh\sqrt{S}\rho / \rho\, Sinh\sqrt{S}\rho^* \tag{6}$$

Our interest is to find the surface temperature, the transform of which is obtained by putting ρ = 1 in equation (6).

$$\therefore \bar{\theta}(1, S) = \bar{\theta}_O(S)$$

$$= \bar{f}(S)\rho^* Sinh\sqrt{S} / Sinh\sqrt{S}\rho^* \tag{7}$$

The inverse transform of above equation gives the required solution. It is very difficult to find the inverse transformation. To proceed further, we write the given function $\bar{f}(S)$ as,

$$\bar{f}(S) = \bar{g}(S)\bar{e}^{\sqrt{S}(1-\rho^*)} \tag{8}$$

We assume that $\bar{g}(S)$ possesses the requisite properties for inverse Laplace transformation. From equations (7) and (8) and after rearrangement, we get

$$\bar{\theta}_O(S) = \bar{g}(S)\rho * \frac{1-e^{-2\sqrt{s}}}{1-e^{-2\sqrt{s}}\rho *} \tag{9}$$

If the interior point is at the center of the sphere, i.e. $\rho* = O$, then equation (9) becomes,

$$\bar{\theta}_O(S) = \bar{g}(S)\frac{(1-e^{-2\sqrt{S}})}{2\sqrt{S}} \tag{10}$$

By taking inverse Laplace transform, the surface temperature $\theta_O(\tau)$ can be determined. The final expressions are of simple form for data given at particular interior points $\rho *$.

When $\rho* = \frac{1}{N}$, where N is an integer, we can write,

$$\frac{1-\bar{e}^{2\sqrt{5}}}{1-\bar{e}^{2\sqrt{s\rho}*}} = 1 + W + W^2 + \dots\dots\dots\dots + W^{N-1} \tag{11}$$

where $w = \bar{e}^{\frac{2}{N}\sqrt{s}}$

For N = 2, i.e. $\rho* = \frac{1}{2} = 0.5$

By taking the inverse Laplace transform, we get

$$\theta_O(\tau) = \frac{1}{2}g(\tau) + \frac{1}{8\sqrt{\pi}}\int_O^\tau \frac{f(\lambda)}{(\tau-\lambda)^{\frac{3}{2}}}e^{-}\frac{1}{16(\tau-\lambda)}d\lambda \tag{12}$$

The second term can be calculated directly from the given temperature data $f(\tau)$. Here $g(\tau)$ is found by the inverse Laplace transformation of equation (8)

For N=3, $\rho* = \frac{1}{3}$, the surface temperature is,

$$\theta_O(\tau) = \frac{1}{3}g(\tau) + \frac{1}{3}f(\tau) + \frac{1}{9\sqrt{\pi}}\int_O^\tau \frac{f(\lambda)}{(\tau-\lambda)^{\frac{3}{2}}}e^{-\frac{1}{9(\tau-\lambda)}}.d\lambda \tag{13}$$

In general, for any $\rho^* = \frac{1}{N}$,

$$\theta_O(\tau) = \frac{1}{N}g(\tau) + \sum_{n=1}^{N-1}\frac{(2n+1-N)}{2\sqrt{\pi}N^2}$$

$$\int_O^\tau \frac{f(\lambda)}{(\tau-\lambda)^{\frac{3}{2}}}.e\frac{-(2n+1-N)^2}{4N^2(\tau-\lambda)}.d\lambda \quad (14)$$

in which terms with (2n + 1 – N) < 0 are replaced by,

$$\frac{n}{N^2\sqrt{\pi}}\int_O^\tau g(\lambda)(\tau-\lambda)^{-\frac{3}{2}}.e^{-n^2/N^2(\tau-\lambda)}.d\lambda \quad (15)$$

while the term with (2n+1-N)=0 is replaced by $f(\tau)/N$. Putting N = 2 and N = 3 in the above equation, we get equations (12) and (13) respectively.

When the interior temperature data $f(\tau)$ is given at any arbitrary position $\rho^*(\neq 0 and \neq \frac{1}{N})$, then the inverse Laplace transform of equation (9), gives

$$\frac{\theta_O(\tau)}{\rho^*} = g(\tau) - \frac{(1+\rho^*)}{2\sqrt{\pi}}\int_O^\tau \frac{f(\lambda)}{(\tau-\lambda)^{\frac{3}{2}}}$$

$$.\bar{e}\frac{(1+\rho^*)^2}{4(\tau-\lambda)}.d\lambda$$

$$+\frac{1}{2\sqrt{\pi}}\sum_{n=1}^{\infty}\int_O^\tau \frac{d\lambda}{(\tau-\lambda)\frac{3}{2}}\left[f(\lambda)(2n\rho^*+\rho^*-1)\right]$$

$$.\bar{e}\frac{(2n\rho^*+\rho^*+1)^2}{4(\tau-\lambda)} - f(\lambda)(2n\rho^*+\rho^*+1).\bar{e}\frac{(2n\rho^*+\rho^*+1)^{\bar{2}}}{4(\tau-\lambda)} \quad (16)$$

when $(2n\rho^*+\rho^*-1)<0$, the first term in the square brackets is replaced by,

$$2n\rho^* g(\lambda)e^{-\frac{(\rho^* n)^2}{(\tau-\lambda)}} \quad (17)$$

For ρ^* near 1, i.e. locations near the surface, the series converges rapidly and few terms can be taken. If $f(\tau)$ data correspond to the center of the sphere, i.e. $\rho^*=0$, the corresponding surface temperature can be obtained from equation (10) as,

$$\theta_O(\tau) = \frac{1}{2\sqrt{\pi}}\left[\int_O^\tau \frac{g(\lambda)}{(\tau-\lambda)^{1/2}}d\lambda - \int_O^\tau \frac{f(\lambda)}{(\tau-\lambda)^{\frac{1}{2}}}.e^{-\frac{1}{4(\tau-\lambda)}}d\lambda\right] \quad (18)$$

It is seen that in most of the terms involving integration, the given temperature data $f(\tau)$ enters directly. For $\rho^* \geq \frac{1}{3}$, $f(\tau)$ alone enters into the integrations, i.e. not g. To carryout integration involving either f - function or the g-function, it is convenient to note,

$$\frac{1}{\sqrt{\pi}}\int_{O}^{\tau}\frac{h(\lambda)}{(\tau-\lambda)^{\frac{3}{2}}}.e^{-\frac{c}{(\tau-\lambda)}}d\lambda$$

$$=\frac{h(\tau)}{\sqrt{C}}\left[1-erf\,(\frac{c}{\tau})^{\frac{1}{2}}\right]$$

$$+\frac{1}{\sqrt{\pi}}\int_{O}^{\tau}\frac{h(\lambda)-h(\tau)}{(\tau-\lambda)^{\frac{3}{2}}}.e^{\frac{c}{(\tau-\lambda)}}d\lambda \qquad (C>0) \tag{19}$$

$$\frac{1}{\sqrt{\pi}}\int_{O}^{\tau}\frac{h(\lambda)}{(\tau-\lambda)^{\frac{3}{2}}}.e^{-\frac{c}{(\tau-\lambda)}}d\lambda$$

$$=2h(\tau)\left\{\left(\frac{\tau}{\pi}\right)^{\frac{1}{2}}\bar{e}^{\frac{c}{\tau}}-C^{\frac{1}{2}}\left[1-erf\,(\frac{c}{\tau})^{\frac{1}{2}}\right]\right\}$$

$$+\frac{1}{\sqrt{\pi}}\int_{O}^{\tau}\frac{h(\lambda)-h(\tau)}{(\tau-\lambda)^{\frac{1}{2}}}e^{-\frac{c}{(\tau-\lambda)}}d\lambda \qquad (c\geq 0)$$

The new integrands involving $[h(\lambda)-h(\tau)]$ are always smaller than the former integrands involving $h(\lambda)$. Therefore, the net effect of this rephrasing is to reduce substantially the contribution of the remaining integrals. This is an especially desirable property if numerical integration techniques are to be applied.

To determine the surface temperature, we have to determine $g(\tau)$. From the inverse Laplace transform of equation (8),we get

$$f(\tau)=\frac{(1-\rho^*)}{2\sqrt{\pi}}\int_{O}^{\tau}\frac{g(\lambda)}{(\tau-\lambda)^{\frac{3}{2}}}\bar{e}\,\frac{(1-\rho^*)^2}{4(\tau-\lambda)}.d\lambda \tag{21}$$

Above equation is an Integral equation for $g(\tau)$ which uses the given temperature history $f(\tau)$ at the interior location ρ^* as input.

To check the accurancy of the method, consider the following problem for which exact solution for the surface temperature is known.

4. Numerical Evaluation

A temperature variation with time was prescribed at the surface and from this, the time – temperature history at an interior location ρ^* was calculated from the direct problem. This give $f(\tau)$. Using this, $g(\tau)$ function is calculated using equation (21) and then $\theta_O(\tau)$ can be determined.

For numerical work, the surface temperature chosen as the initial input as linear function, i.e. $\theta_O(\tau) = \tau$, and the interior point is chosen as $\rho^* = 0.5$. The results of calculations are shown in Figure. The solid line corresponds to the exact variation of the surface temperature with time.

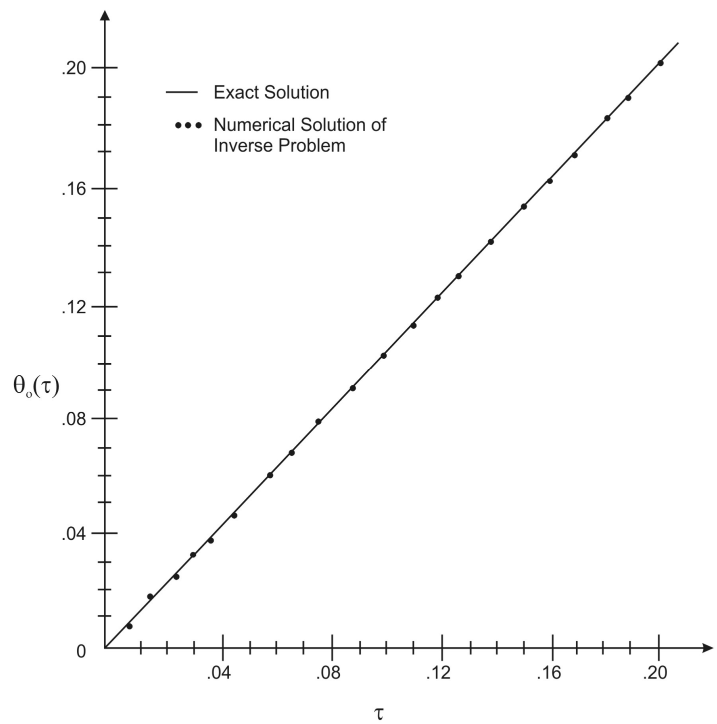

Fig : Surface Temperature Results for the Sphere

The discrete points represent values calculated using the present method of analysis for the inverse problem, i.e. from equation (5.12). By taking $\Delta\tau = 0.01$ (step size) the calculations were carried out for 20 points. In general, the discrete points are different from the exact solution. From the figure it can be observed that the numerical solutions of inverse problem is almost equals to exact solution.

REFERENCES

1. Beck, J.V, "Calculation of Heat flux from and internal Temperature History", ASME Trans, Journal of Heat Transfer, Vol.46, 1962.

2. Blackwell.B.F, "An Efficient Technique for the Numerical Solution of the One Dimensional Inverse Problems of Heat Conduction", Num.Heat transfer, 4, 1981.

3. Burggraf.O.R, "An Exact Solution of Inverse Problem of Heat Conduction", J.of Heat Transfer, 86c, 1964.

4. Imber.M, "Two dimensional Inverse Heat Conduction Problem"AIAAJ,13,1975.

5. Langford .D, "Analytic Solutions of the One Dimensional Heat Equation", Q.Appl.Math, 24, 1967 .

6. Mulholland.G.P, "Diffusion through Composite Media", Int.J.of Heat and Mass Transfer.Vol.15, 1972

7. Stolz.G, "Numerical Solution to an Inverse Problems of Heat Conduction", Journal of Heat transfer, Trans.ASME, Vol.82, Series. C, 1960.

8. Williams's and Curry D.M "An Analytical Study of Surface Heat flux Determinations. of Space craft", 14,1977.