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THE STABILITY OF A FOUR SPECIES: A PREY-PREDATOR-HOST-COMMENSAL-SYN ECO-SYSTEM-IV (Both the Hosts are washed out states)

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ABSTRACT: This paper deals with an investigation on a Four Species Syn-Ecological System (Both the Hosts are washed out states). The System comprises of a Prey (S_1), a Predator (S_2) that survives upon S_1, two Hosts S_3 and S_4 for which S_1, S_2 are commensal respectively i.e., S_3 and S_4 benefit S_1 and S_2 respectively, without getting effected either positively or adversely. Further S_3 and S_4 are neutral. The model equations of the system constitute a set of four first order non-linear ordinary differential coupled equations. In all, there are sixteen equilibrium points. Criteria for the asymptotic stability of three of the sixteen equilibrium points : Both the Hosts are washed out states only are established in this paper. The system would be stable if all the characteristic roots are negative, in case they are real, and have negative real parts, in case they are complex. The linearized equations for the perturbations over the equilibrium points are analyzed to establish the criteria for stability and the trajectories illustrated.

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1. INTRODUCTION:

Mathematical modeling of Eco-System was initiated in 1925 by Lotka [10] and in 1931 by Volterra[14]. The general concepts of modeling have been presented in the treatises of Meyer[11], Kushing[7], Kapur J.N. [5,6] and several others. The ecological interactions can be broadly classified as Prey-Predator, Commensalism, Competition, Neutralism, Mutualism and so on. N.C. Srinivas [13] studied competitive eco-systems of two species and three species with limited and unlimited resources. Later Lakshminarayan [8], Lakshminarayan and Pattabhi Ramacharyulu [9] studied Prey-Preadtor ecological models with partial cover for the Prey and alternate food for the Predator. Recently, Archana Reddy [1] and Bhaskara Rama Sharma [2] investigated diverse problems related to two species competitive systems with time delay, employing analytical and numerical techniques. Further Phani Kumar, Seshagiri Rao and Pattabhi Ramacharyulu [12] studied the stability of a Host-A flourishing commensal species pair with limited resources. The present authors Hari Prasad.B and Pattabhi Ramacharyulu.N.Ch studied the stability of the fully washed out state [3] and co-existent state [4]. Continuation of this criteria for the stability of the Host (S_3) of S_1 and Host (S_4) of S_2 only washed out states of the system are presented in this paper.

2. BASIC EQUATIONS OF THE MODEL:

Notation Adopted:

S_1 : Prey for S_2 and commensal for S_3.

S_2 : Predator surviving upon S_1 and commonsal for S_4.

S_3 : Host for the commonsal – Prey (S_1).

S_4 : Host of the commonsal – Predator (S_2)

$N_1(t)$: The Population of the Prey (S_1)

$N_2(t)$: The Population of the Predator (S_2)

$N_3(t)$: The Population of the Host (S_3) of the Prey (S_1)

$N_4(t)$: The Population of the Host (S_4) of the Predator (S_2)

t : Time instant

a_1,a_2,a_3,a_4 : Natural growth rates of S_1, S_2, S_3, S_4

$a_{11},a_{22},a_{33},a_{44}$: Self inhibition coefficients of S_1, S_2, S_3, S_4

a_{12},a_{21} : Interaction (Prey-Predator) coefficients of S_1 due to S_2 and S_2 due to S_1

a_{13} : Coefficient for commensal for S_1 due to the Host S_3

a_{24} : Coefficient for commensal for S_2 due to the Host S_4

$\frac{a_1}{a_{11}},\frac{a_2}{a_{22}},\frac{a_3}{a_{33}},\frac{a_4}{a_{44}}$: Carrying capacities of S_1, S_2, S_3, S_4

Further the variables N_1, N_2, N_3, N_4 are non-negative and the model parameters a_1, a_2, a_3, a_4; a_{11}, a_{22}, a_{33}, a_{44}; a_{12}, a_{21}, a_{13}, a_{24} are assumed to be non-negative constants.

The model equations for the growth rates of S_1, S_2, S_3, S_4 are

$$\frac{dN_1}{dt} = a_1 N_1 - a_{11} N_1^2 - a_{12} N_1 N_2 + a_{13} N_1 N_3 \quad \ldots. \quad (2.1)$$

$$\frac{dN_2}{dt} = a_2 N_2 - a_{22} N_2^2 + a_{21} N_1 N_2 + a_{24} N_2 N_4 \quad \ldots. \quad (2.2)$$

$$\frac{dN_3}{dt} = a_3 N_3 - a_{33} N_3^2 \quad \ldots. \quad (2.3)$$

$$\frac{dN_4}{dt} = a_4 N_4 - a_{44} N_4^2 \quad \ldots. \quad (2.4)$$

3 EQUILIBRIUM STATES:

The system under investigation has sixteen equilibrium states defined by

$$\frac{dN_i}{dt} = 0,\ i = 1,2,3,4 \qquad \text{………… (3.1)}$$

are given in the following table.

S.No.	Equilibrium States	Equilibrium Point
1	Fully Washed out state	$\overline{N_1} = 0, \overline{N_2} = 0, \overline{N_3} = 0, \overline{N_4} = 0$
2	Only the Host (S_4)of S_2 survives	$\overline{N_1} = 0, \overline{N_2} = 0, \overline{N_3} = 0, \overline{N_4} = \frac{a_4}{a_{44}}$
3	Only the Host (S_3)of S_1 survives	$\overline{N_1} = 0, \overline{N_2} = 0, \overline{N_3} = \frac{a_3}{a_{33}}, \overline{N_4} = 0$
4	Only the Predator S_2 survives	$\overline{N_1} = 0, \overline{N_2} = \frac{a_2}{a_{22}}, \overline{N_3} = 0, \overline{N_4} = 0$
5	Only the Prey S_1 survives	$\overline{N_1} = \frac{a_1}{a_{11}}, \overline{N_2} = 0, \overline{N_3} = 0, \overline{N_4} = 0$
6	Prey (S_1) and Predator (S_2) washed out	$\overline{N_1} = 0, \overline{N_2} = 0, \overline{N_3} = \frac{a_3}{a_{33}}, \overline{N_4} = \frac{a_4}{a_{44}}$
7	Prey (S_1) and Host (S_3) of S_1 washed out	$\overline{N_1} = 0, \overline{N_2} = \frac{a_2 a_{44} + a_4 a_{24}}{a_{22} a_{44}}, \overline{N_3} = 0, \overline{N_4} = \frac{a_4}{a_{44}}$
8	Prey (S_1) and Host (S_4) of S_2 washed out	$\overline{N_1} = 0, \overline{N_2} = \frac{a_2}{a_{22}}, \overline{N_3} = \frac{a_3}{a_{33}}, \overline{N_4} = 0$
9	Predator (S_2) and Host (S_3) of S_1 washed out	$\overline{N_1} = \frac{a_1}{a_{11}}, \overline{N_2} = 0, \overline{N_3} = 0, \overline{N_4} = \frac{a_4}{a_{44}}$
10	Predator (S_2) and Host (S_4) of S_2 washed out	$\overline{N_1} = \frac{a_1 a_{33} + a_3 a_{13}}{a_{11} a_{13}}, \overline{N_2} = 0, \overline{N_3} = \frac{a_3}{a_{33}}, \overline{N_4} = 0$
11	Prey (S_1) and Predator (S_2)survives	$\overline{N_1} = \frac{a_1 a_{22} - a_2 a_{12}}{a_{11} a_{22} + a_{12} a_{21}}, \overline{N_2} = \frac{a_1 a_{21} + a_2 a_{11}}{a_{11} a_{22} + a_{12} a_{21}}, \overline{N_3} = 0, \overline{N_4} = 0$
12	Only the Prey (S_1) washed out	$\overline{N_1} = 0, \overline{N_2} = \frac{a_2 a_{44} + a_4 a_{24}}{a_{22} a_{44}}, \overline{N_3} = \frac{a_3}{a_{33}}, \overline{N_4} = \frac{a_4}{a_{44}}$
13	Only the predator (S_2) washed out	$\overline{N_1} = \frac{a_1 a_{23} + a_3 a_{13}}{a_{11} a_{13}}, \overline{N_2} = 0, \overline{N_3} = \frac{a_3}{a_{33}}, \overline{N_4} = \frac{a_4}{a_{44}}$

14	Only the Host (S_3) of S_1 washed out	$\overline{N_1}=\frac{\delta_2}{\delta_1},\overline{N_2}=\frac{\delta_3}{\delta_1},\overline{N_3}=0,\overline{N_4}=\frac{a_4}{a_{44}}$ *where* $\delta_1=a_{44}(a_{11}a_{22}+a_{12}a_{21})>0$ $\delta_2=a_1a_{22}a_{44}-a_{12}(a_2a_{44}+a_4a_{24})$ $\delta_3=a_1a_{21}a_{44}-a_{11}(a_2a_{44}+a_4a_{24})$
15	Only the Host (S_4) of S_2 washed out	$\overline{N_1}=\frac{\sigma_2}{\sigma_1},\overline{N_2}=\frac{\sigma_3}{\sigma_1},\overline{N_3}=\frac{a_3}{a_{33}},\overline{N_4}=0$ *where* $\sigma_1=a_{33}(a_{11}a_{22}+a_{12}a_{21})>0$ $\sigma_2=a_{22}(a_1a_{33}+a_3a_{13})-a_2a_{12}a_{33}$ $\sigma_3=a_{21}(a_1a_{33}+a_3a_{13})+a_2a_{11}a_{33}>0$
16	The co-existent state (or) Normal steady state	$\overline{N_1}=\frac{a_{22}a_{44}\psi_1-a_{12}a_{33}\psi_2}{\psi_3},\overline{N_2}=\frac{a_{21}a_{44}\psi_1+a_{11}a_{33}\psi_2}{\psi_3},$ $\overline{N_3}=\frac{a_3}{a_{33}},\overline{N_4}=\frac{a_4}{a_{44}}$ *where* $\psi_1=a_1a_{33}+a_3a_{13}>0$ $\psi_2=a_2a_{44}+a_4a_{24}>0$ $\psi_3=a_{33}a_{44}(a_{11}a_{22}+a_{12}a_{21})>0$

The present paper deals with the Host (S_3) of S_1 and Host (S_4) of S_2 are washed out states only. The stability of the other equilibrium states will be presented in the forth coming communications.

4. Stability of the Host (S_3) of S_1 and Host (S_4) of S_2 washed out equilibrium states (Sl. Nos 4,5,11 in the above table)

4.1 Equilibrium point $\overline{N}_1=0,\ \overline{N}_2=\frac{a_2}{a_{22}},\ \overline{N}_3=0,\ \overline{N}_4=0$:

Let us consider small deviations $u_1(t),\ u_2(t),\ u_3(t),\ u_4(t)$ from the steady state

i.e., $N_i(t)=\overline{N}_i+u_i(t),\ i=1,2,3,4$ ……(4.1.1)

where $u_i(t)$ is a small perturbations in the species S_i.

Substituting (4.1.1) in (2.1), (2.2), (2.3), (2.4) and neglecting products and higher powers of $u_1,\ u_2,\ u_3,\ u_4$

we get

$$\frac{du_1}{dt} = r_1 u_1 \quad .....(4.1.2), \qquad \frac{du_2}{dt} = \frac{a_2 a_{21}}{a_{22}} u_1 - a_2 u_2 + \frac{a_2 a_{24}}{a_{22}} u_4 \quad ....(4.1.3)$$

$$\frac{du_3}{dt} = a_3 u_3 \quad ....(4.1.4), \qquad \frac{du_4}{dt} = a_4 u_4 \quad ....(4.1.5)$$

where $r_1 = a_1 - \frac{a_2 a_{12}}{a_{22}}$ ......(4.1.6)

The characteristic equation of which is

$$(\lambda + r_1)(\lambda + a_2)(\lambda - a_3)(\lambda - a_4) = 0 \quad .....(4.1.7)$$

Case (A): When $r_1 > 0$ (i.e., when $\frac{a_1}{a_2} > \frac{a_{12}}{a_{22}}$)

The roots r_1, a_3, a_4 are positive and $-a_2$ is negative

Hence the steady state is **unstable.**

The solutions of the equations (4.1.1) (4.1.2), (4.1.3), (4.1.4) are

$$u_1 = u_{10} e^{r_1 t} \quad ...... \quad (4.1.8)$$

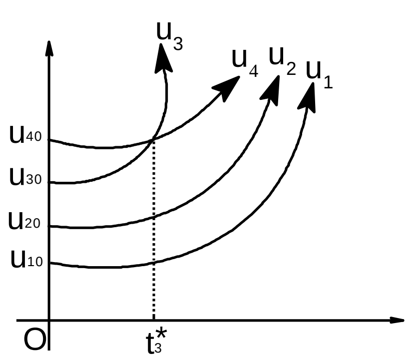
$$u_2 = (u_{20} - m - n) e^{-a_2 t} + m e^{r_1 t} + n e^{a_4 t} \quad ...... \quad (4.1.9)$$

$$u_3 = u_{30} e^{a_3 t} \quad ....(4.1.10), \qquad u_4 = u_{40} e^{a_4 t} \quad ...... \quad (4.1.11)$$

Where $m = \frac{a_2 a_{21} u_{10}}{a_{22}(r_1 + a_2)}$, $n = \frac{a_2 a_{24} u_{40}}{a_{22}(a_4 + a_2)}$ ...... (4.1.12)

There would arise in all 576 cases depending upon the ordering of the magnitudes of the growth rates a_1, a_2, a_3, a_4 and the initial values of the perturbations $u_{10}(t)$, $u_{20}(t)$, $u_{30}(t)$, $u_{40}(t)$ of the species S_1, S_2, S_3, S_4. Of these 576 situations some typical variations are illustrated through respective solution curves that would facilitate to make some reasonable observations.

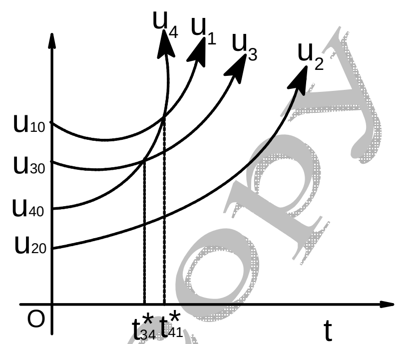
Case (i): If $u_{10} < u_{20} < u_{30} < u_{40}$ and $r_1 < a_2 < a_4 < a_3$



In this case the Prey(S_1) has the least natural birth rate. Initially the Host (S_4) of S_2 dominates over the Host (S_3) of S_1 till the time instant t^*_{34} and there after the dominance is reversed. The time t^*_{34} may be called the dominance time of the Host (S_4) of S_2 over the host (S_3) of S_1.

Here $t^*_{34} = \frac{1}{a_3 - a_4} \log\left(\frac{u_{40}}{u_{30}}\right)$ .....(4.1.13)

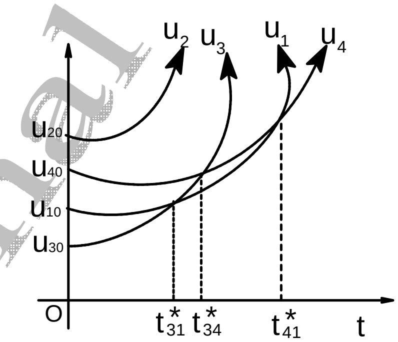
Case (ii): If $u_{20} < u_{40} < u_{30} < u_{10}$ and $a_2 < a_3 < r_1 < a_4$.



In this case the Predator (S_2) has the least natural birth rate. Initially the Prey (S_1) dominates over the Host (S_4) of S_2 till the time instant t^*_{41} and there after the dominance is reversed. Also the host (S_3) of S_1 dominates over the Host (S_4) of S_2 till the time instant and the dominance gets reversed there after.

Here $t^*_{41} = \dfrac{1}{r_1 - a_4} \log\left(\dfrac{u_{40}}{u_{10}}\right)$ …..(4.1.14)

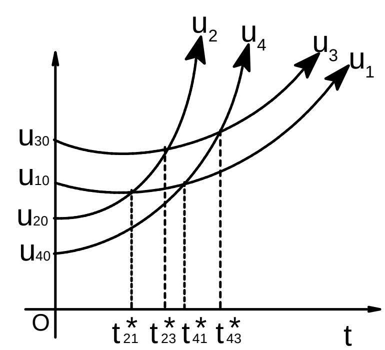
Case (iii) : If $u_{30} < u_{10} < u_{40} < u_{20}$ and $a_4 < r_1 < a_2 < a_3$



In this case the Host (S_4) of S_2 has the least natural birth rate. Initially it is dominated over by the Prey (S_1), Host (S_3) of S_1 till the time instant t^*_{41}, t^*_{34} respectively and there after the dominance is reversed. Also the Prey (S_1) dominates over its Host (S_3) till the time instant t^*_{31} and the dominance gets reversed there after.

Here $t^*_{31} = \dfrac{1}{r_1 - a_3} \log\left(\dfrac{u_{30}}{u_{10}}\right)$ …..(4.1.15)

Case (iv): If $u_{40} < u_{20} < u_{10} < u_{30}$ and $r_1 < a_3 < a_4 < a_2$



In this case the Prey (S_1) has the least natural birth rate. Initially it is dominated over by the Predator (S_2), Host (S_4) of N_2 till the time instant t^*_{21}, t^*_{41} respectively and there after the dominance is reversed. Also the Host (S_3) of S_1 dominates over the Predator (S_2), Host (S_4) of S_2 till the time instant t^*_{23}, t^*_{43} respectively and the dominance is gets reversed there after.

4.1.A. Trajectories of perturbations:

The trajectories in the $u_1 - u_3$ plane given by

$$\left(\frac{u_1}{u_{10}}\right)^{a_3} = \left(\frac{u_3}{u_{30}}\right)^{r_1} \quad \text{……(4.1.16)}$$

and are showing in fig. 1

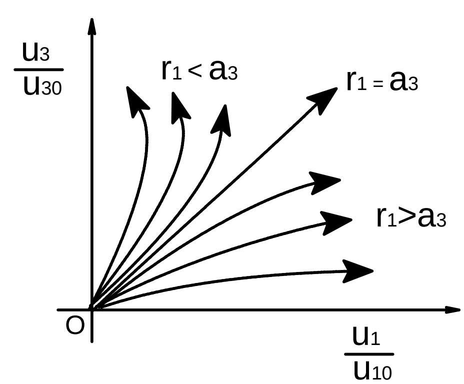


Fig. 1

Also the trajectories in the $u_1 - u_4$, $u_3 - u_4$ planes are

$$\left(\frac{u_1}{u_{10}}\right)^{a_4} = \left(\frac{u_4}{u_{40}}\right)^{r_1}, \left(\frac{u_3}{u_{30}}\right)^{a_4} = \left(\frac{u_4}{u_{40}}\right)^{a_3} \text{ respectively} \qquad \ldots\ldots(4.1.17)$$

and the trajectories in the $u_1 - u_2$, $u_2 - u_3$, $u_2 - u_4$ planes are

$$y = Px^{\frac{-a_2}{r_1}} + Qx + Rx^{\frac{a_4}{r_1}}, \quad y = Px_1^{\frac{-a_2}{a_3}} + Qx_1^{\frac{r_1}{a_3}} + Rx_1^{\frac{a_4}{a_3}} \qquad \ldots\ldots(4.1.18)$$

$$y = Px_2^{\frac{-a_2}{a_4}} + Qx_2^{\frac{r_1}{a_4}} + Rx_2 \text{ respectively} \qquad \ldots\ldots(4.1.19)$$

Where $x = \frac{u_1}{u_{10}}, \; y = \frac{u_2}{u_{20}}, \; x_1 = \frac{u_3}{u_{30}}, \; x_2 = \frac{u_4}{u_{40}}$ ……(4.1.20)

and $P = \frac{u_{20} - m - n}{u_{20}}, \; Q = \frac{m}{u_{20}}, \; R = \frac{n}{u_{20}}$ ……(4.1.21)

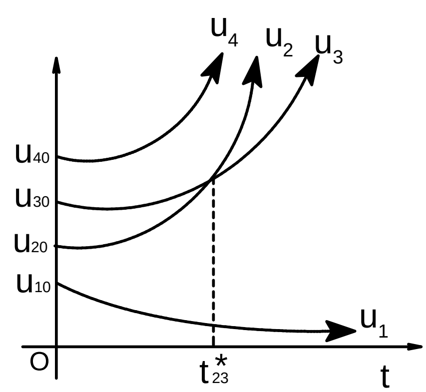
Case (B): When $r_1 < 0$ (ie, when $\frac{a_1}{a_2} < \frac{a_{12}}{a_{22}}$)

The roots a_3, a_4 are positive and $r_1, -a_2$ are negative. Hence the study state in **unstable**.

In this case the solutions are same as in case (A)

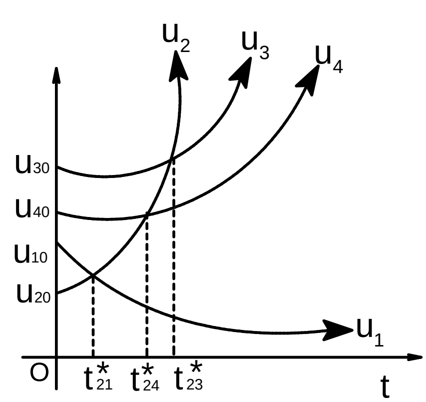
Case (i) : If $u_{10} < u_{20} < u_{30} < u_{40}$ and $a_3 < r_1 < a_2 < a_4$

In this case the Prey (S_1) has the least natural birth rate. Initially the Host (S_3) of S_1 dominates over the Predator (S_2) till the time instant t^*_{23} and there after the dominance is reversed.



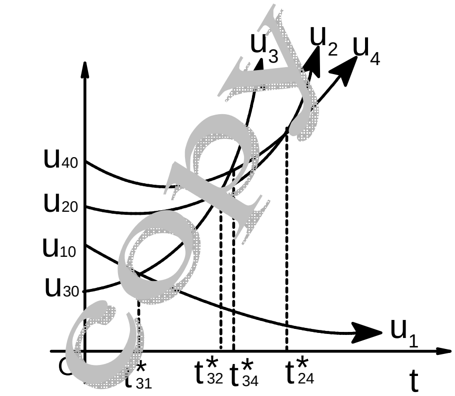
Case (ii): If $u_{20} < u_{10} < u_{40} < u_{30}$ and $r_1 < a_4 < a_3 < a_2$

In this case the Prey (S_1) has the least natural birth rate. Initially the Prey (S_1), Host (S_4) of S_2, Host (S_3) of S_1 dominates over the Predator (S_2) till the time instant t^*_{21}, t^*_{24}, t^*_{23} respectively and there after the dominance is reversed.



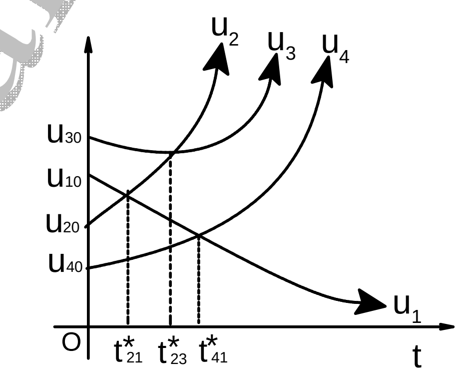
Case (iii): If $u_{30} < u_{10} < u_{20} < u_{40}$ and $a_4 < r_1 < a_2 < a_3$

In this case the Prey (S_1) has the least natural birth rate. Initially the Prey (S_1), Predator (S_2), Host (S_4) of S_2 dominates over the Host (S_3) of S_1 till the time instant t^*_{31}, t^*_{32}, t^*_{34} respectively and thereafter the dominance is reversed. Also the Host (S_4) of S_2 dominates over the Predator S_2 till the time instant t^*_{34} and the dominance gets reversed there after.



Case (iv) : If $u_{40} < u_{20} < u_{10} < u_{30}$ and $a_4 < r_1 < a_3 < a_2$

In this case the Prey (S_1) has the least natural birth rate. Initially it is dominated over by the Predator (S_2), Host (S_4) of S_2 till the time instant t^*_{21}, t^*_{41} respectively and there after the dominance is reversed. Also the Host (S_3) of S_1 dominates over the Predator (S_2) till the time instant t^*_{23} and the dominance gets reversed there after.



4.1.B:Trajectories of perturbations:

The trajectories in the $u_1 - u_3$ plane given by

$$\left(\frac{u_1}{u_{10}}\right)^{a_3} = \left(\frac{u_3}{u_{30}}\right)^{r_1} \quad \text{…. (4.1.22)}$$

and are shown in Fig. 2

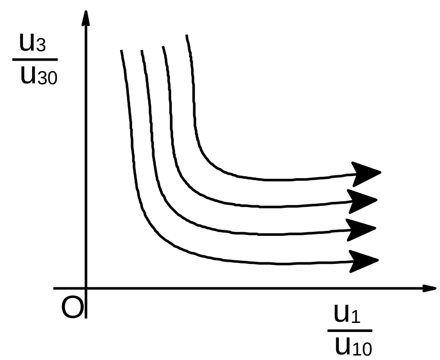


Fig. 2

Also the trajectories in the $u_1 - u_4$, $u_3 - u_4$ planes are

$$\left(\frac{u_1}{u_{10}}\right)^{a_4} = \left(\frac{u_4}{u_{40}}\right)^{r_1}, \left(\frac{u_3}{u_{30}}\right)^{a_4} = \left(\frac{u_4}{u_{40}}\right)^{a_3} \text{ respectively} \quad \text{……(4.1.23)}$$

and the trajectories in the $u_1 - u_2$, $u_2 - u_3$, $u_2 - u_4$ planes

are $y = Px^{\frac{-a_2}{r_1}} + Qx + Rx^{\frac{a_4}{r_1}}$, $y = Px_1^{\frac{-a_2}{a_3}} + Qx_1^{\frac{r_1}{a_3}} + Rx_1^{\frac{a_4}{a_3}}$ …….. (4.1.24)

$y = Px_2^{\frac{-a_2}{a_4}} + Qx_2^{\frac{r_1}{a_4}} + Rx_2$ respectively …… (4.1.25)

4.2. Equilibrium point $\overline{N}_1 = \frac{a_1}{a_{11}}, \; \overline{N}_2 = 0, \; \overline{N}_3 = 0, \; \overline{N}_4 = 0$:

Substituting (4.1.1) in (2.1), (2.2), (2.3), (2.4) and neglecting products and higher powers of u_1, u_2, u_3, u_4 we get

$$\frac{du_1}{dt} = -a_1 u_1 - \frac{a_1 a_{12}}{a_{11}} u_2 + \frac{a_1 a_{13}}{a_{11}} u_3 \qquad \text{…..(4.2.1)}$$

$$\frac{du_2}{dt} = l_2 u_2 \quad \text{…..(4.2.2)}, \qquad \frac{du_3}{dt} = a_3 u_3 \qquad \text{……(4.2.3)}$$

$$\frac{du_4}{dt} = a_4 u_4 \qquad \text{…..(4.2.4)}$$

Where $l_2 = \left(a_2 + \frac{a_1 a_{21}}{a_{11}} \right) > 0$ ……(4.2.5)

The characteristic equation of which is

$$(\lambda + a_1)(\lambda - l_2)(\lambda - a_3)(\lambda - a_4) = 0 \qquad \text{……(4.2.6)}$$

The roots l_2, a_3, a_4 are positive and $-a_1$ is negative.

Hence the steady state is **unstable.**

The solutions of the equations (4.2.1), (4.2.2), (4.2.3) (4.2.4) are

$$u_1 = (u_{10} + m_1 - n_1)e^{-a_1 t} - m_1 e^{-l_2 t} + n_1 e^{a_3 t} \qquad \text{….(4.2.7)}$$

$$u_2 = u_{20} e^{l_2 t} \quad \text{…. (4.2.8)}, \qquad u_3 = u_{30} e^{a_3 t} \quad \text{…(4.2.9)}$$

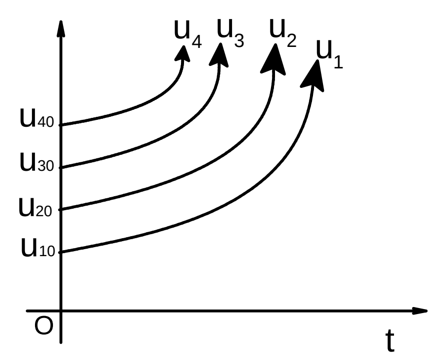
$$u_4 = u_{40} e^{a_4 t} \qquad \text{….(4.2.10)}$$

Here $m_1 = \frac{a_1 a_{12} u_{20}}{a_{11}(l_2 + a_1)} > 0$, $n_1 = \frac{a_1 a_{13} u_{30}}{a_{11}(a_3 + a_1)} > 0$ ….(4.2.11)

and $l_2 = \frac{a_2}{a_1} + \frac{a_{21}}{a_{11}} > 0$ …. (4.2.12)

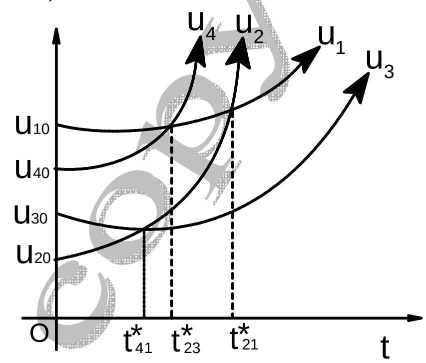
Case (i): If $u_{10} < u_{20} < u_{30} < u_{40}$ and $a_1 < l_2 < a_3 < a_4$

In this case the Prey (S_1) has the least natural birth rate and the Host (S_4) of S_2 dominates the prey (S_1) Predator (S_2), Host (S_3) of S_1 in natural growth rate as well as in its population strength.



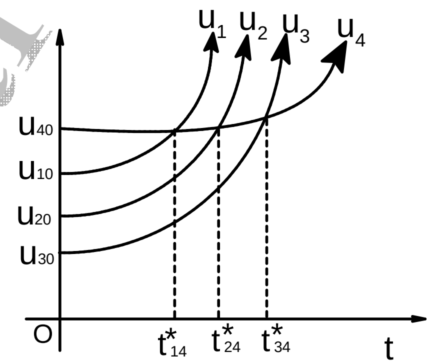
Case (ii): If $u_{20} < u_{30} < u_{40} < u_{10}$ and $a_3 < a_1 < l_2 < a_4$

In this case the Host (S_3) of S_1 has the least natural birth rate. Initially it is dominated over by the Predator (S_2) till the time instant t^*_{23} and there after the dominance is reversed. Also the Prey (S_1) dominates over the Predator (S_2), Host (S_4) of S_2 till the time instant t^*_{21}, t^*_{41} respectively and the dominance gets reversed there after.



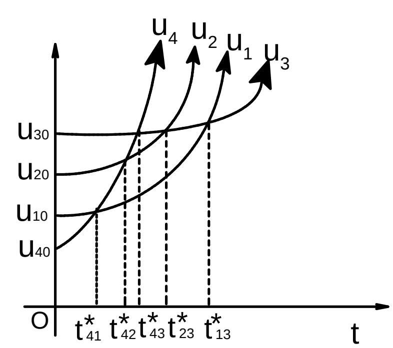
Case (iii) : If $u_{30} < u_{20} < u_{10} < u_{40}$ and $a_4 < a_3 < l_2 < a_1$

In this case the Host (S_4) of S_2 has the least natural birth rate. Initially it is dominated over by the Prey (S_1), Predator (S_2), Host (S_3) of S_1 till the time instant t^*_{14}, t^*_{24}, t^*_{34} respectively and there after the dominance is reversed.



Case (iv): If $u_{40} < u_{10} < u_{20} < u_{30}$ and $a_3 < a_1 < l_2 < a_4$

In this case the Host (S_3) of S_1 has the least natural birth rate. Initially it is dominated over by the Host (S_4) of S_2, Predator (S_2) , Prey (S_1) till the time instant t^*_{43} , t^*_{23} , t^*_{13} respectively and thereafter the dominance is reversed. Also the Predator (S_2), Prey (S_1) dominates over the Host (S_4)of S_2 till the time instant t^*_{42} , t^*_{41} respectively and the dominance gets reversed there after.



Trajectories of perturbations:

The trajectories in the $u_2 - u_3$ plane given by

$$\left(\frac{u_2}{u_{20}}\right)^{a_3} = \left(\frac{u_3}{u_{30}}\right)^{l_2} \qquad \text{……(4.2.13)}$$

and are shown in fig. 3

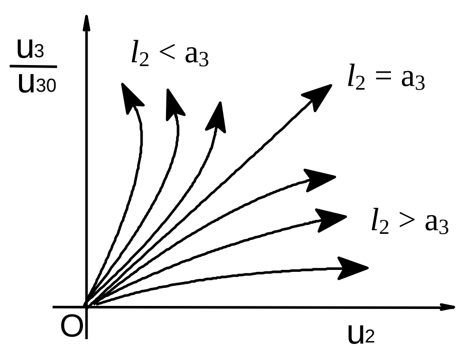


Fig. 3

Also the trajectories in the $u_2 - u_4$, $u_3 - u_4$, $u_1 - u_2$, $u_1 - u_3$, $u_1 - u_4$ planes are

$$\left(\frac{u_2}{u_{20}}\right)^{a_4} = \left(\frac{u_4}{u_{40}}\right)^{l_2}, \left(\frac{u_3}{u_{30}}\right)^{a_4} = \left(\frac{u_4}{u_{40}}\right)^{a_3}$$

….(4.2.14)

$$x = P_1 y^{-a_1/l_2} - Q_1 y + R_1 y^{\frac{a_3}{l_2}}, \quad x = P_1 x_1^{\frac{-a_1}{a_3}} - Q_1 x_1^{\frac{l_2}{a_3}} + R_1 x_1 \qquad \text{….(4.2.15)}$$

$$x = P_1 x_2^{-\frac{a_1}{a_4}} - Q_1 x_2^{l_2/a_4} + R_1 x_2^{a_3/a_4} \text{ respectively} \qquad \text{….. (4.2.16)}$$

where

$$P_1 = \frac{u_{10} + m_1 - n_1}{u_{10}}, \; Q_1 = \frac{m_1}{u_{10}}, \; R_1 = \frac{u_1}{u_{10}} \qquad \text{….. (4.2.17)}$$

4.3. Equilibrium point: $\overline{\mathbf{N}}_1 = \dfrac{\mathbf{a_1 a_{22} - a_2 a_{12}}}{\mathbf{a_{11} a_{22} + a_{12} a_{21}}}$, $\overline{\mathbf{N}}_2 = \dfrac{\mathbf{a_1 a_{21} + a_2 a_{11}}}{\mathbf{a_{11} a_{22} + a_{12} a_{21}}}$ $\overline{\mathbf{N}}_3 = \mathbf{0}$, $\overline{\mathbf{N}}_4 = \mathbf{0}$:

Substituting (4.1.4) in (2.1), (2.2), (2.3), (2.4) and neglecting products and higher powers of u_1, u_2, u_3, u_4

we get

$$\frac{du_1}{dt} = \mu_1 u_1 - \frac{a_{12}\mu_2}{\mu} u_2 + \frac{a_{13}\mu_2}{\mu} u_3 \qquad \text{…..(4.3.1)}$$

$$\frac{du_2}{dt} = \frac{a_{12}\gamma_1}{\mu} u_1 + \gamma_2 u_2 + \frac{a_{24}\gamma_1}{\mu} u_4 \qquad \text{…..(4.3.2)}$$

$$\frac{du_3}{dt} = a_3 u_3 \quad \text{…..(4.3.3)}, \quad \frac{du_4}{dt} = a_4 u_4 \qquad \text{….(4.3.4)}$$

where $\mu_1 = a_1 - \dfrac{2a_{11}\mu_2}{\mu} - \dfrac{a_{12}\gamma_1}{\mu}$ …..(4.3.5)

$$\mu_2 = a_1 a_{22} - a_2 a_{12} \; ; \; \gamma_1 = a_1 a_{21} + a_2 a_{11} > 0 \qquad \text{……(4.3.6)}$$

$$\gamma_2 = a_2 - \frac{2a_{22}\gamma_1}{\mu} + \frac{a_{21}\mu_2}{\mu}, \; \mu = a_{11} a_{22} + a_{12} a_{21} > 0 \qquad \text{……(4.3.7)}$$

The characteristic equation of which is

$$\left[\lambda^2 - (\mu_1 + \gamma_2)\lambda - \frac{a_{12} a_{21} \mu_2 \gamma_1}{\mu^2}\right](\lambda - a_3)(\lambda - a_4) = 0 \qquad \text{…..(4.3.8)}$$

Two of the four roots a_3 and a_4 are positive.

Hence the steady state is **unstable.**

Let λ_1, λ_2 be the zeros of the quadratic polynomial on the L.H.S. of the equation (4.3.8)

Case (A): When the roots λ_1 and λ_2 have opposite signs

The solutions of the equations (4.3.1), (4.3.2), (4.3.3), (4.3.4) are

$$u_1 = \left[\frac{a_{12}\mu_2(\beta - u_{20}) - (\alpha - u_{10})(\mu_1 - \lambda_2)\mu}{\mu(\lambda_1 - \lambda_2)}\right]e^{\lambda_1 t}$$

$$+\left[\frac{a_{12}\mu_2(\beta - u_{20}) - (\alpha - u_{10})(\mu_1 - \lambda_1)\mu}{\mu(\lambda_2 - \lambda_1)}\right]e^{\lambda_2 t} + Ae^{a_3 t} - Be^{a_4 t} \quad \ldots\ldots(4.3.9)$$

$$u_2 = \left[\frac{a_{12}\mu_2(\beta - u_{20}) - (\alpha - u_{10})(\mu_1 - \lambda_2)\mu}{a_{12}\mu_2(\lambda_1 - \lambda_2)}\right](\mu_1 - \lambda_1)e^{\lambda_1 t}$$

$$+\left[\frac{a_{12}\mu_2(\beta - u_{20}) - (\alpha - u_{10})(\mu_1 - \lambda_1)}{a_{12}\mu_2(\lambda_2 - \lambda_1)}\right](\mu_1 - \lambda_2)e^{\lambda_2 t}$$

$$+\left[\frac{\mu A}{a_{12}\mu_2}(\mu_1 - a_3) + \frac{a_{13}}{a_{12}}u_{30}\right]e^{a_3 t} + \frac{\mu B}{a_{12}\mu_2}(a_4 - \mu_1)e^{a_4 t} \quad \ldots\ldots\ldots(4.3.10)$$

$$u_3 = u_{30}e^{a_3 t} \quad \ldots\ldots\ (4.3.11), \qquad u_4 = u_{40}e^{a_4 t} \quad \ldots\ldots\ (4.3.12)$$

where

$$A = \frac{\overline{A}}{a_3^2 - (\mu_1 + \gamma_2)a_3 + C},\ \ B = \frac{\overline{B}}{a_4^2 - (\mu_1 + \gamma_2)a_4 + C} \quad \ldots\ldots\ldots\ (4.3.13)$$

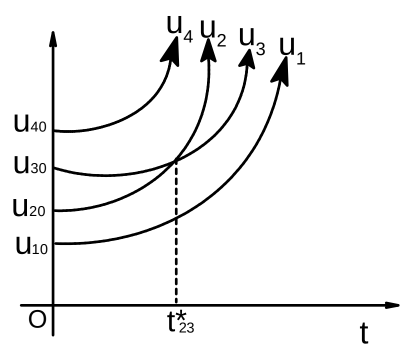
$$\overline{A} = \frac{a_{13}\mu_2 u_{30}}{\mu}(a_3 - \gamma_2),\ \overline{B} = \frac{a_{12}a_{24}\gamma_1\mu_2}{\mu}u_{40} \quad \ldots\ldots\ldots(4.3.14)$$

$$C = m_1\gamma_2 + \frac{a_{12}a_{21}\gamma_1\mu_2}{\mu^2},\ \ \alpha = A - B \quad \ldots\ldots\ldots(4.3.15)$$

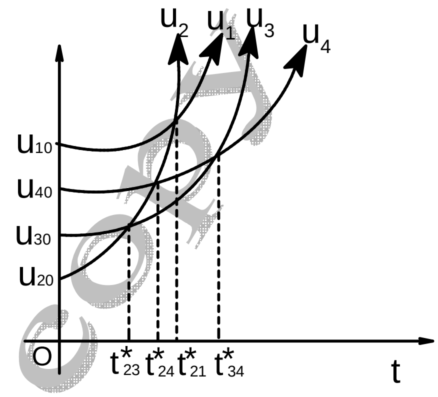
$$\beta = \frac{a_{13}u_{30}}{a_{12}} + \frac{\mu}{a_{12}\mu_2}\left[A(\mu_1 - a_3) + B(a_4 - \mu_1)\right] \quad \ldots\ldots\ldots\ldots\ldots\ (4.3.16)$$

Case (i): If $u_{10} < u_{20} < u_{30} < u_{40}$ and $a_4 < a_3 < a_1 < a_2$

In this case the Prey (S_1) has the least natural birth rate. Initially the Host (S_3) of S_1 dominates over the Predator (S_2) till the time instant t_{23}^* and thereafter the dominance is reversed.

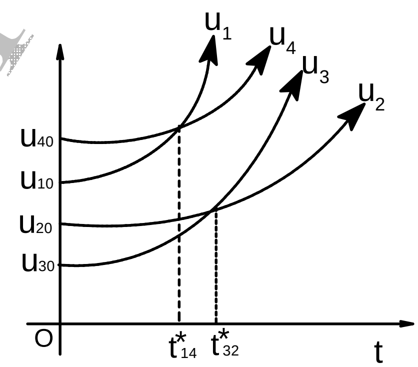


Case(ii): If $u_{20} < u_{30} < u_{40} < u_{10}$ and $u_4 < u_3 < u_1 < u_2$



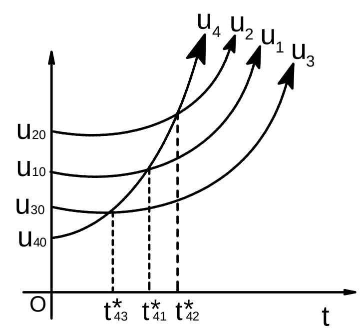
In this case the Host (S_4) of S_2 has the least natural birth rate. Initially it is dominated over by the Predator (S_2), Host (S_3) of S_1 till the time instant t^*_{24}, t^*_{34} respectively and thereafter the dominance is reversed. Also the Host (S_3) of S_1 dominates over the Predator (S_2) till the time instant t^*_{23} and thereafter the dominance is reversed. Similarly the Prey (S_1) dominates over the Predator (S_2) till the time instant t^*_{21} and the dominance gets reversed there after.

Case(iii): If $u_{30} < u_{20} < u_{10} < u_{40}$ and $a_2 < a_3 < a_4 < a_1$



In this case the Predator (S_2) has the least natural birth rate. Initially it is dominated over by the Host (S_3) of S_1 till the time instant t^*_{32} and thereafter the dominance is reversed. Also the Host (S_4) of S_2 dominates over the Prey (S_1) till the time instant t^*_{14} and dominance gets reversed thereafter.

Case (iv): If $u_{40} < u_{30} < u_{10} < u_{20}$ and $a_3 < a_1 < a_2 < a_4$

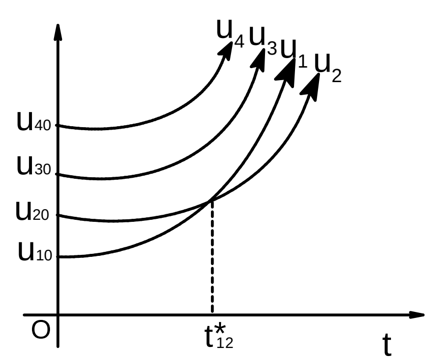


In this case the Host (S_3) has the least natural birth rate. Initially it is dominated over by the Host (S_4) of S_2 till the time instant t^*_{43} and thereafter the dominance is reversed. Also the Prey (S_1), Predator (S_2) dominates over the Host (S_4) of S_2 till the time instant t^*_{41}, t^*_{42} respectively and the dominance gets reversed thereafter.

Case (B) : When the roots λ_1 and λ_2 have same signs

In this case the solutions are same as in case (A).

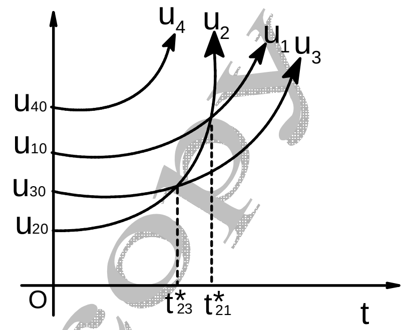
Case(i): If $u_{10} < u_{20} < u_{30} < u_{40}$ and $a_2 < a_1 < a_3 < a_4$



In this case the Predator (S_2) has the least natural birth rate. Initially it is dominated over by the Prey (S_1) till the time instant t^*_{12} and thereafter the dominance is reversed.

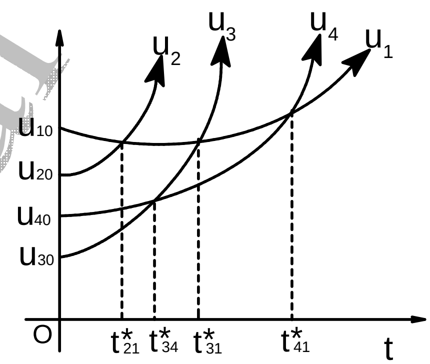
Case (ii): If $u_{20} < u_{30} < u_{10} < u_{40}$ and $a_3 < a_1 < a_2 < a_4$

In this case the Host (S_3) of S_1 has the least natural birth rate. Initially it is dominated over by the Predator (S_2) till the time instant t^*_{23} and thereafter the dominance is reversed. Also the Prey (S_1) dominates over the Predator (S_2) till the time instant t^*_{21} and the dominance gets reversed thereafter.



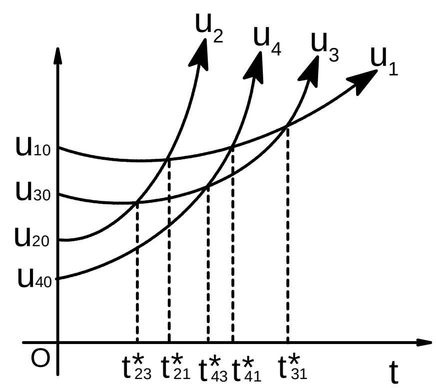
Case (iii) : If $u_{30} < u_{40} < u_{20} < u_{10}$ and $a_1 < a_4 < a_3 < a_2$

In this case the Prey (S_1) has least natural birth rate. Initially it is dominated over by the Predator (S_2), Host (S_3) of S_1 , Host (S_4) of S_2 till the time instant t^*_{21}, t^*_{31}, t^*_{41} respectively and thereafter the dominance is reversed. Also the Host (S_4) of S_2 dominates over the Host (S_3) of S_1 till the time instant t^*_{34} and the dominance gets reversed there after.



Case(iv): If $u_{40} < u_{20} < u_{30} < u_{40}$ and $a_1 < a_3 < a_4 < a_2$

In this case the Prey (S_1) has the least natural birth rate. Initially it is dominated over by the Predator (S_2), Host (S_4) of S_2, Host (S_3) of S_1 till the time instant t^*_{21}, t^*_{41}, t^*_{31} respectively and thereafter the dominance is reversed. Also the Host (S_3) of S_1 dominates over the Predator (S_2), Host (S_4) of S_2 till the time instant t^*_{23}, t^*_{43} respectively and the dominance gets reversed thereafter.



Trajectories of Perturbations:

The trajectories in the $u_3 - u_4$ plane given by

$$\left(\frac{u_3}{u_{30}}\right)^{a_4} = \left(\frac{a_4}{a_{40}}\right)^{a_3} \qquad \ldots(4.3.17)$$

and are shown in Fig. 4

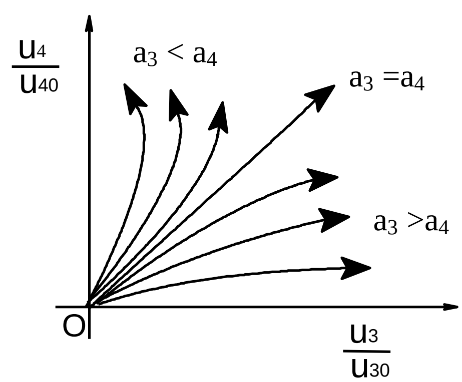


Fig. 4

And the trajectories in the $u_1 - u_3, u_1 - u_4,\ u_2 - u_3,\ u_2 - u_4$ planes are

$$x = A_1 x_1^{\frac{\lambda_1}{a_3}} + B_1 x_1^{\frac{\lambda_2}{a_3}} + C_1 x_1 - D_1 x_1^{\frac{a_4}{a_3}} \quad \text{……..(4.3.18)}$$

$$x = A_1 x_2^{\frac{\lambda_1}{a_4}} + B_1 x_2^{\frac{\lambda_2}{a_4}} + C_1 x_2^{\frac{a_3}{a_4}} - D_1 x_2 \quad \text{……(4.3.19)}$$

$$y = A_2 x_1^{\frac{\lambda_1}{a_3}} + B_2 x_1^{\frac{\lambda_2}{a_3}} + C_2 x_1 - D_2 x_1^{\frac{a_4}{a_3}} \quad \text{……(4.3.20)}$$

$$y = A_2 x_2^{\frac{\lambda_1}{a_4}} + B_2 x_2^{\frac{\lambda_2}{a_4}} + C_2 x_2^{\frac{a_3}{a_4}} - D_2 x_2 \quad \text{…..(4.3.21)}$$

Where

$$A_1 = \frac{a_{12}\mu_2(\beta - u_{20}) - (\alpha - u_{10})(\mu_1 - \lambda_2)\mu}{u_{10}\mu(\lambda_1 - \lambda_2)} \quad \text{…..(4.3.22)}$$

$$B_1 = \frac{a_{12}\mu_2(\beta - u_{20}) - (\alpha - u_{10})(\mu_1 - \lambda_1)\mu}{u_{10}\mu(\lambda_2 - \lambda_1)} \quad \text{……(4.3.23)}$$

$$C_1 = \frac{A}{u_{10}},\ \ D_1 = \frac{B}{u_{10}} \quad \text{……(4.3.24)}$$

$$A_2 = \left[\frac{a_{12}\mu_2(\beta - u_{20}) - (\alpha - u_{10})(\mu_1 - \lambda_2)}{u_{20}\mu_2 a_{12}(\lambda_1 - \lambda_2)}\right](\mu_1 - \lambda_1) \quad \text{…….(4.3.25)}$$

$$B_2 = \left[\frac{a_{12}\mu_2(\beta - u_{20}) - (\alpha - u_{10})(\mu_1 - \lambda_1)}{u_{20}\mu_2 a_{12}(\lambda_2 - \lambda_1)}\right](\mu_1 - \lambda_2) \quad \text{…….(4.3.26)}$$

$$C_2 = \frac{1}{u_{20}}\left[\frac{\mu A}{a_{12}\mu_2}(\mu_1 - a_3) + \frac{a_{13}}{a_{12}}u_{30}\right] \quad \text{…….(4.3.27)}$$

$$D_2 = \frac{\mu B}{u_{20}a_{12}\mu_2}(a_4 - \mu_1) \quad \text{…….(4.3.28)}$$

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