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A model of two mutually interacting species with unlimited resources for both the species

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Abstract

The present paper deals with an analytical investigation of a model of two species mutually interacting with resources for both the species being unlimited. The model is characterized by a coupled system of first order non-linear ordinary differential equations. Only one equilibrium point is identified and its stability criteria are derived. It is observed, in case when the death rate of the second species is greater than its birth rate, there exist only one equilibrium point. Stability of the equilibrium point and the solutions for the linearized perturbed equations are obtained. However when the death rate is greater than the birth rate for both the species, there exists two equilibrium points. We derived their stability criteria and obtained the solutions of the linearized perturbed equations.

1. Introduction

Ever since research in the discipline of theoretical ecology was initiated by Lotka [8] and by Volterra [12], several mathematicians and ecologists contributed in the growth of this area of knowledge as has been extensively reported in the treatises of Meyer [9], Cushing [2], Paul Colinvaux [10], Freedman [3], Kapur [5, 6]. The ecological interactions can be broadly classified as prey-predation, competition, mutualism and so on. N.C. Srinivas [11] studied the competitive eco-systems of two species and three species with regard to limited and unlimited resources. Later, Lakshmi Narayan [7] has investigated the two species prey-predator models. Recently stability analysis of competitive species was investigated by Archana Reddy [1]. Mutualism is any relationship between two species of organisms that benefits both species. Pollination (flowers and insects), seed dispersal (berries and fruit eaten by birds and animals), and lichens (fungus and algae) are examples for mutualism.

The present investigation is devoted to the analytical study of a model of two mutually interacting species with unlimited resources for both the species. The model is characterized by a coupled pair of first order non-linear ordinary differential equations. The equilibrium points of the system are identified and the stability analysis is carried out.

Before describing a model, first we make the following assumptions:
N_1 is the population of the first species, N_2, the population of the second species, a_1, a_2 are respectively the rates of natural growth of the first and second species, α_{12} is the rate of increase of the first species due to interaction with the second species, α_{21} is the rate of increase of the second species due to interaction with the first species. Further note that the variables N_1, N_2 and the model parameters a_1, a_2, α_{12}, α_{21} are non-negative. If the death rate is greater than the birth rate for any species, we continue to use the same notation as natural growth rate with negative sign for the rate of difference. The model equations for a two species mutualising are given by a system of non-linear ordinary differential equations.

2. Basic Equations:

The equation for the growth rate of first species (N_1) is given by

$$\frac{dN_1}{dt} = a_1 N_1 + \alpha_{12} N_1 N_2 \tag{2.1}$$

The equation for the growth rate of second species (N_2) is given by

$$\frac{dN_2}{dt} = a_2 N_2 + \alpha_{21} N_1 N_2 \tag{2.2}$$

The equilibrium states are given by

$$\frac{dN_1}{dt} = 0 \text{ and } \frac{dN_2}{dt} = 0$$

That is

$$N_1\{a_1 + \alpha_{12} N_2\} = 0 \quad \text{and} \quad N_2\{a_2 + \alpha_{21} N_1\} = 0 \tag{2.3}$$

A solution $(\bar{N}_1, \bar{N}_2)$ of (2.3) is called the equilibrium state of (2.1)-(2.2).
The system under investigation has one equilibrium state given by

$$\bar{N}_1 = 0; \bar{N}_2 = 0. \tag{2.4}$$

In this state both the species are washed out.

Now we study the stability of the equilibrium state. Let us write

$$N = (N_1, N_2) = \bar{N} + U$$

where $U = (u_1, u_2)$ is a small perturbation over the equilibrium state $\bar{N} = (\bar{N}_1, \bar{N}_2)$.
The basic equations (2.1), (2.2) are linearized to obtain the system for the perturbed state,

$$\frac{dU}{dt} = AU \tag{2.5}$$

where

$$A = \begin{bmatrix} a_1 + \alpha_{12}\bar{N}_2 & \alpha_{12}\bar{N}_1 \\ \alpha_{21}\bar{N}_2 & a_2 + \alpha_{21}\bar{N}_1 \end{bmatrix} \tag{2.6}$$

The characteristic equation for the system is

$$det\left[A - \lambda I\right] = 0 \tag{2.7}$$

The equilibrium state is stable, if both the roots of the equation (2.7) are negative in case they are real or have negative real parts in case they are complex.

To discuss the stability of equilibrium state $\overline{N}_1 = 0$; $\overline{N}_2 = 0$, we consider small perturbations $u_1(t)$ and $u_2(t)$ from the steady state.

That is, we write

$$N_1 = \overline{N}_1 + u_1(t), \tag{2.8}$$

$$N_2 = \overline{N}_2 + u_2(t) \tag{2.9}$$

Substituting (2.8) and (2.9) in (2.1) and (2.2), we get

$$\frac{du_1}{dt} = a_1 u_1 + \alpha_{12} u_1 u_2$$

$$\frac{du_2}{dt} = a_2 u_2 + \alpha_{21} u_1 u_2$$

After linearization, we get

$$\frac{du_1}{dt} = a_1 u_1 \tag{2.10}$$

and

$$\frac{du_2}{dt} = a_2 u_2 \tag{2.11}$$

The characteristic equation is

$$(\lambda - a_1)(\lambda - a_2) = 0,$$

whose roots a_1, a_2 are both positive. Hence the equilibrium state is unstable.

The solutions of equations (2.10) and (2.11) are

$$u_1 = u_{10}\ e^{a_1 t} \tag{2.12}$$

$$u_2 = u_{20}\ e^{a_2 t} \tag{2.13}$$

where u_{10}, u_{20} are the initial values of u_1 and u_2. The solution curves are illustrated in figures 1 to 4.

Case 1: $a_1 < a_2$ and $u_{10} < u_{20}$ i.e. the second species dominates the first species in the natural growth rate as well as in its initial population strength.

In this case, the second species continues to dominate the first species as shown in fig.1.

Case 2: $a_1 < a_2$ and $u_{10} > u_{20}$ i.e. the second species dominates the first species in the natural growth rate but its initial strength is less than that of first species.

In this case, the first species out numbers the second species till the time-instant,

$$t = t^* = \frac{ln\ \{u_{10} / u_{20}\}}{(a_2 - a_1)}$$

after that the second species out numbers the first species. This is illustrated in figure 2

Case 3: $a_1 > a_2$ and $u_{10} < u_{20}$ i.e. the first species dominates the second species in the natural growth rate but its initial strength is less than that of the second species.

Here the second species out numbers the first species till the time-instant,

$$t = t^* = \frac{ln\ \{u_{10} / u_{20}\}}{(a_2 - a_1)}$$

after that the first species out numbers the second species. This is illustrated in figure 3

Case 4: $a_1 > a_2$ and $u_{10} > u_{20}$ i.e. the first species dominates the second species in the natural growth as well as in its initial population strength.
Clearly the first species continues to be out numbering the second species as shown in figure 4.

We derive the trajectories of the perturbed species. The trajectories in the (u_1, u_2) plane are given by

$$\left[\frac{u_1}{u_{10}}\right]^{a_2} = \left[\frac{u_2}{u_{20}}\right]^{a_1}$$

and these are illustrated in figure 5.

3. The death rate of the second species is greater than its birth rate.

Under this the basic equations are

$$\frac{dN_1}{dt} = a_1 N_1 + \alpha_{12} N_1 N_2 \tag{3.1}$$

$$\frac{dN_2}{dt} = -a_2 N_2 + \alpha_{21} N_1 N_2 \tag{3.2}$$

The system under investigation has one equilibrium state given by

$$\overline{N}_1 = 0; \overline{N}_2 = 0.$$

This is a state where both the species are washed out.

We discuss the stability of equilibrium state when $\overline{N}_1 = 0$; $\overline{N}_2 = 0$.

Consider small perturbations $u_1(t)$ and $u_2(t)$ from the steady state.
That is, we write

$$N_1 = \overline{N}_1 + u_1(t), \tag{3.3}$$

$$N_2 = \overline{N}_2 + u_2(t). \tag{3.4}$$

Substituting (3.3) and (3.4) in (3.1) and (3.2), we get

$$\frac{du_1}{dt} = a_1 u_1 + \alpha_{12} u_1 u_2$$

$$\frac{du_2}{dt} = -a_2 u_2 + \alpha_{21} u_1 u_2$$

After linearization, we get

$$\frac{du_1}{dt} = a_1 u_1 \tag{3.5}$$

and

$$\frac{du_2}{dt} = -a_2 u_2 \tag{3.6}$$

The characteristic equation is

$$(\lambda - a_1)(\lambda + a_2) = 0$$

One root of this equation is $\lambda_1 = a_1$ which is positive and the other root is $\lambda_2 = -a_2$ which is negative. Hence the equilibrium state is **unstable.**
The solutions of equations (3.5) and (3.6) are

$$u_1 = u_{10}\ e^{a_1 t} \tag{3.7}$$

$$u_2 = u_{20}\ e^{-a_2 t} \tag{3.8}$$

where u_{10}, u_{20} are the initial values of u_1 and u_2. The solution curves are illustrated in figures 6 and 7.

Case 1: $u_{10} > u_{20}$ i.e. initially the first species dominates the second species.

We notice that the first species is going away from the equilibrium point while the second species approaches asymptotically to the equilibrium point. Hence the state is unstable.

Case 2: $u_{10} < u_{20}$ i.e. initially the second species dominates the first species.

Initially when the second species out numbers the first species, the domination of the second species over the first continues till the time,

$$t = t^* = \frac{ln\ \{u_{20} / u_{10}\}}{(a_1+a_2)}$$

after that the first species out numbers the second species and grows indefinitely while the second species approaches asymptotically to the equilibrium point. Hence the state is unstable.

We shall now describe the trajectories of the perturbed species. The trajectories in the (u_1,u_2) plane are given by

$$\left[\frac{u_1}{u_{10}}\right]^{a_2} = \left[\frac{u_2}{u_{20}}\right]^{-a_1}$$

4. The death rate is greater than the birth rate for both the species

The basic equations governing the system are

$$\frac{dN_1}{dt} = -a_1 N_1 + \alpha_{12} N_1 N_2 \tag{4.1}$$

$$\frac{dN_2}{dt} = -a_2 N_2 + \alpha_{21} N_1 N_2 \tag{4.2}$$

Here we come across two equilibrium states:

I. $\overline{N}_1 = 0; \overline{N}_2 = 0$,

the state in which both the species are washed out.

II. $\overline{N}_1 = \frac{a_2}{\alpha_{21}}$; $\overline{N}_2 = \frac{a_1}{\alpha_{12}}$,

the state in which both the species co-exist.

<u>Equilibrium state I</u> (fully washed out state):

To discuss the stability of equilibrium state $\overline{N}_1 = 0$; $\overline{N}_2 = 0$, we consider small perturbations $u_1(t)$ and $u_2(t)$ from the steady state, i.e. we write

$$N_1 = \overline{N}_1 + u_1(t), \tag{4.3}$$

$$N_2 = \overline{N}_2 + u_2(t). \tag{4.4}$$

Substituting (4.3) and (4.4) in (4.1) and (4.2), we get

$$\frac{du_1}{dt} = -a_1 u_1 + \alpha_{12} u_1 u_2$$

$$\frac{du_2}{dt} = -a_2 u_2 + \alpha_{21} u_1 u_2$$

After linearization, we get

$$\frac{du_1}{dt} = -a_1 u_1 \quad (4.5)$$

$$\frac{du_2}{dt} = -a_2 u_2 \quad (4.6)$$

The characteristic equation is

$$(\lambda + a_1)(\lambda + a_2) = 0$$

The roots of this equation, $\lambda_1 = -a_1$ and $\lambda_2 = -a_2$ are both negative. Hence the equilibrium state is **stable.**

The solutions of equations (4.5) and (4.6) are

$$u_1 = u_{10}\, e^{-a_1 t} \quad (4.7)$$

$$u_2 = u_{20}\, e^{-a_2 t} \quad (4.8)$$

where u_{10}, u_{20} are the initial values of u_1 and u_2. The solution curves are illustrated in figures 8 to 11.

CASE 1: $a_1 > a_2$ and $u_{10} > u_{20}$ i.e. the first species dominates the second species in the natural growth rate as well as in its initial population strength.

In this case the first species continues out numbering the second species as shown in figure 8. It is evident that both the species converging asymptotically to the equilibrium point. Hence the state is stable.

CASE 2: $a_1 > a_2$ and $u_{10} < u_{20}$ i.e. the first species dominates the second species in the natural growth rate but its initial strength is less than that of second species.

In this case, initially the second species out numbers the first species and this continues till the time,

$$t = t^* = \frac{ln\,\{u_{10} / u_{20}\}}{(a_1 - a_2)}$$

after that the first species out numbers the second species. As t $\rightarrow \infty$ both u_1 and u_2 approach to the equilibrium point. Hence the state is stable.

CASE 3: $a_1 < a_2$ and $u_{10} < u_{20}$ i.e. the second species dominates the first species in the natural growth rate as well as in its initial population strength.

In this case the second species always out numbers the first species. It is evident that both the species converging asymptotically to the equilibrium point. Hence the state is **stable**.

CASE 4: $a_1 < a_2$ and $u_{10} > u_{20}$ i.e. the second species dominates the first species in the natural growth rate but its initial strength is less than that of first species.

In this case, initially the first species out numbers the second species and this continues up to the time,

$$t = t^* = \frac{ln\,\{u_{10} / u_{20}\}}{(a_1 - a_2)}$$

there after the second species out numbers the first species. As t $\to \infty$ both u_1 and u_2 approach to the equilibrium point. Hence the state is stable. Also the trajectories in the (u_1, u_2) plane are given by

$$\left[\frac{u_1}{u_{10}}\right]^{-a_2} = \left[\frac{u_2}{u_{20}}\right]^{-a_1}$$

<u>Equilibrium state II</u> (coexistence state):

We have

$$\overline{N}_1 = \frac{a_2}{\alpha_{21}}; \ \overline{N}_2 = \frac{a_1}{\alpha_{12}}$$

Substituting (4.3) and (4.4) in (4.1) and (4.2), we get

$$\frac{du_1}{dt} = \alpha_{12}\overline{N}_1 u_2 + \alpha_{12} u_1 u_2$$

$$\frac{du_2}{dt} = \alpha_{21}\overline{N}_2 u_1 + \alpha_{21} u_1 u_2$$

After linearization, we get

$$\frac{du_1}{dt} = \alpha_{12}\overline{N}_1 u_2 \qquad (4.9)$$

$$\frac{du_2}{dt} = \alpha_{21}\overline{N}_2 u_1 \qquad (4.10)$$

The characteristic equation is

$$\lambda^2 - \alpha_{12}\alpha_{21}\overline{N}_1\overline{N}_2 = 0$$

That is

$$\lambda^2 - a_1 a_2 = 0$$

One root of this equation is $\lambda_1 = \sqrt{a_1 a_2}$ which is positive and the other root is $\lambda_2 = -\sqrt{a_1 a_2}$ which is negative. Hence the equilibrium state is **unstable**.

The trajectories are given by

$$u_1 = \left[\frac{u_{10}\lambda_1 + u_{20}\alpha_{12}\overline{N}_1}{\lambda_1 - \lambda_2}\right] e^{\lambda_1 t} + \left[\frac{u_{10}\lambda_2 + u_{20}\alpha_{12}\overline{N}_1}{\lambda_2 - \lambda_1}\right] e^{\lambda_2 t} \qquad (4.11)$$

$$u_2 = \left[\frac{u_{20}\lambda_1 + u_{10}\alpha_{21}\overline{N}_2}{\lambda_1 - \lambda_2}\right] e^{\lambda_1 t} + \left[\frac{u_{20}\lambda_2 + + u_{10}\alpha_{21}\overline{N}_2}{\lambda_2 - \lambda_1}\right] e^{\lambda_2 t} \qquad (4.12)$$

Figures 12 & 13 illustrate the behaviour of the species u_1 and u_2

Case 1: For $u_{10} > u_{20}$, we have

In this case, the first species is noted to be going away from the equilibrium point while the second species asymptotically approaches to the equilibrium point. Hence the state is unstable.

Case 2: If $u_{10} < u_{20}$, we have

In this case the second species dominates the first species till the time,

$$t = t^* = \frac{1}{\lambda_2 - \lambda_1} \ ln \left[\frac{(b_2 - \lambda_1)u_{10} + (\lambda_1 - b_1)u_{20}}{(b_2 - \lambda_2)u_{10} + (\lambda_2 - b_1)u_{20}}\right]$$

where

$$b_1 = \alpha_{12}\overline{N}_1\,;\; b_2 = \alpha_{21}\overline{N}_2$$

and there after the first species dominates the second species and grows indefinitely while the second species approaches asymptotically to the equilibrium point. Hence the state is unstable. Also the trajectories in the (u_1,u_2) plane are given by

$$[u_2^{(a-1)(v_1-v_2)}]d = \frac{(u_1-u_2v_1)^{av_1}}{(u_1-v_2u_2)^{av_2}}$$

Where v_1 and v_2 are roots of the quadratic equation $av^2+c=0$ with $a=\alpha_{21}\overline{N}_2$; $c=-\alpha_{12}\overline{N}_1$ and d is an arbitrary constant.

6. Trajectories:

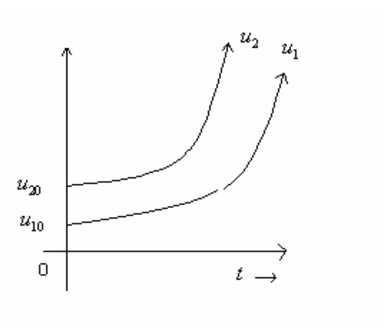

Fig. 1

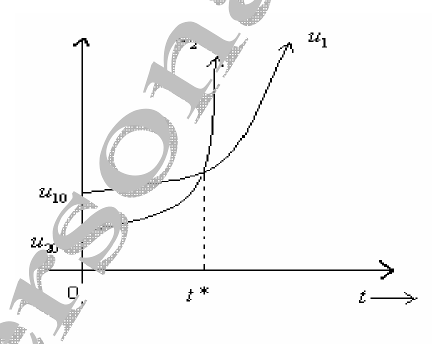

Fig. 2

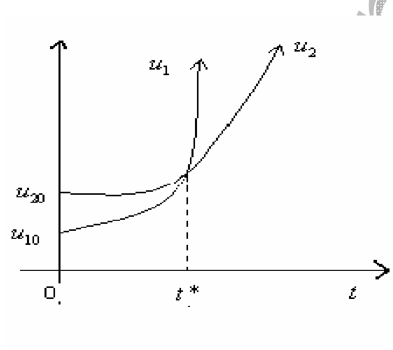

Fig. 3

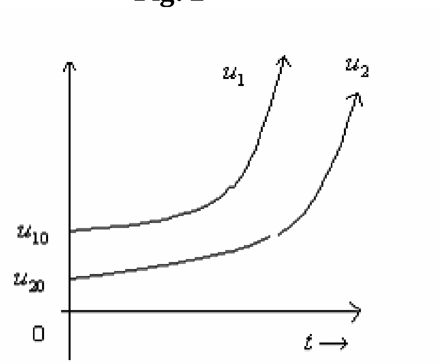

Fig. 4

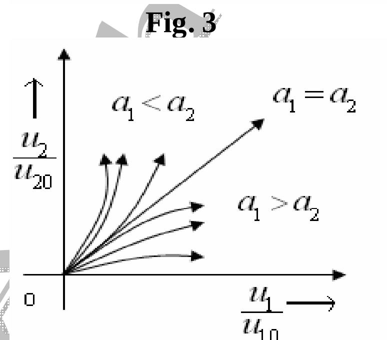

Fig. 5

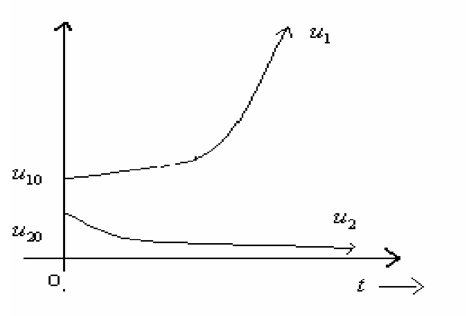

Fig. 6

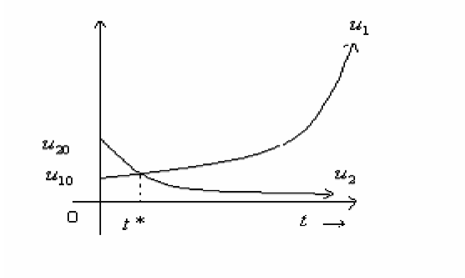

Fig. 7

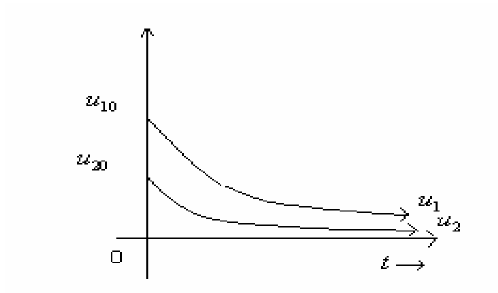

Fig. 8

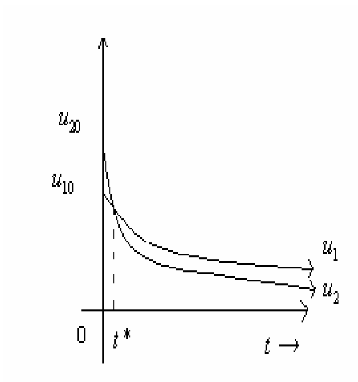

Fig. 9

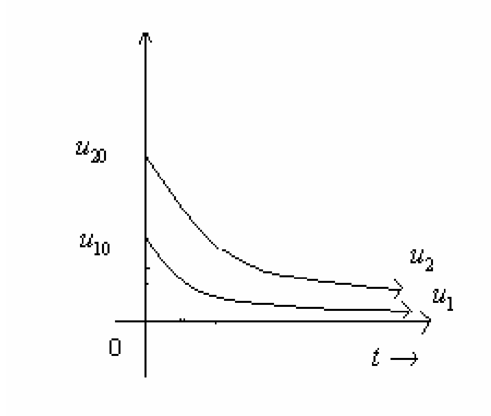

Fig. 10

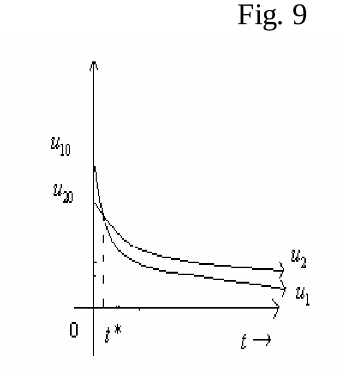

Fig. 11

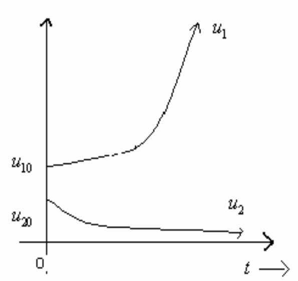

Fig. 12

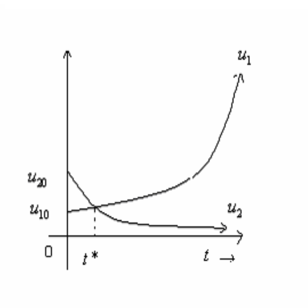

Fig. 13

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