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STABILITY OF A SYN- ECOSYSTEM CONSISTING OF A PREY-PREDATOR AND HOST COMMENSAL TO THE PREY (WITH MORTALITY RATE FOR THE PREY)

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ABSTRACT:

The present paper deals with a three species ecosystem consisting of a prey (N_1), a predator (N_2) and a host (N_3) commensal to the prey having a mortality rate. The mathematical model equations constitute a set of three first order non-linear simultaneous equations in N_1, N_2 and N_3. The equation for host is non-linear but de-coupled with the prey-predator pair. In all, there will be six equilibrium points of the model and the criteria for their stability are discussed. The trajectories of perturbations over the equilibrium states on species have been drawn.

1. INTRODUCTION:

Mathematical modeling of ecosystems was initiated by Lotka [9] and Volterra [16]. The general concepts of modeling have been presented in the treatises of Meyer [10], Cushing [3], Paul Colinvaux [11], Freedman [4], Kapur [5, 6] and several others. The ecological symbiosis can be broadly classified as prey-predator, competition, mutualism, commensalism and so on. N.C.Srinivas [15] studied the competitive ecosystems of two species and three species with limited and unlimited recourses. Lakshminarayana and Pattabhi Ramacharyulu [7, 8] investigated prey-predator ecological models with a partial cover for the prey and alternative food for the predator and prey predator model with cover for prey and alternate food for the predator and to me delay . Recently, stability analysis of competitive species was carried out by Archana Reddy, Pattabhi Ramacharyulu and Gandhi [1], by Bhaskara Rama Sarma and Pattabhi Ramacharyulu [2].

While the mutualism between two species was examined by Ravindra Reddy [13]. Recently Phanikumar ,SeshagiriRao and Pattabhi Ramacharyuly [12] studied on the stability of a host- A flourishing commensal species pair with limited resources. SeshagiriRao, Phanikumar and Pattabhi Ramacharyuly [14] investigated on the stability of a host- A decaying commensal species pair with limited resources.

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The present investigation is an analytical study of three species system:
Prey-Predator and host system. In all, six equilibrium points are identified based on the model equations and these are spread over three distinct classes.
(i)Fully washed out (ii).semi/partially washed out and (iii) co-existent states. Criteria for the asymptotic stability of the states have been derived. It is noticed that all the states are stable except in the following cases.

(i).Fully washed out state.
(ii).The prey and predator washed out but not the host.
(iii).The prey and host washed out but not the predator.

Some threshold diagrams have also been drawn to highlight the regions of the asymptotic **stability/instability**.

2. NOTATION ADOPTED:

N_1: The population of the prey-commensal.
N_2: The population of the predation striving of the prey N_1.
N_3: The population of the host to the prey N_1.
d_1: The natural death/decay rate of N_1.
a_2: The natural growth rate of N_2.
a_3: The natural growth rate of N_3.
a_{ii}: The rate of decrease of N_i due to insufficient resources of N_i, i =1,2,3.
a_{12}: The decrease of prey (N_1) due to inhibition by the predator (N_2).
a_{13}: The rate of increase of the commensal (N_1) due to its successful promotion by the host (N_3).
a_{21}: The rate of increase of the predator (N_2) due to its successful attacks on the Prey (N_1).
$e_1\left(= d_1 / a_{11}\right)$ extinction coefficient of prey (N_1).
$k_2\left(= a_2 / a_{22}\right)$ is the carrying capacity of predator (N_2).
$k_3\left(= a_3 / a_{33}\right)$ is the carrying capacity of host (N_3).
$p\left(= a_{12} / a_{11}\right)$ is the coefficient of prey-commensal inhibition of the predator..
$q\left(= a_{21} / a_{22}\right)$ is the coefficient predation consumption of the prey.
$c\left(= a_{13} / a_{11}\right)$ is the coefficient of commensalism.
The commensalism is strong or weak according as $c \geq e_1$ or $c < e_1$.

3. BASIC BALANCE EQUATIONS OF THE MODEL:

The model equations for a three species multi-reactive ecosystem are given by the following system of non-linear ordinary differential equations.

1. Equation for the growth rate of the prey commensal species (N_1):

$$\frac{dN_1}{dt} = a_{11} N_1 \left[- e_1 - N_1 - pN_2 + cN_3\right] \qquad \text{………………} (3.1)$$

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2. Equation for the growth rate of predator species (N_2):

$$\frac{dN_2}{dt} = a_{22} N_2 [k_2 - N_2 + qN_1] \qquad \ldots\ldots\ldots\ldots (3.2)$$

3. Equation for the growth rate of host species (N_3):

$$\frac{dN_3}{dt} = a_{33} N_3 [k_3 - N_3] \qquad \ldots\ldots\ldots\ldots (3.3)$$

Further the variables N_1, N_2, N_3 are non-negative and the model parameters d_1, a_2, a_3, a_{11}, a_{12}, a_{13}, a_{21}, a_{22}, a_{33} are non-negative and assumed to be constants.

4. EQUILIBRIUM STATES:

These are given by $\frac{dN_i}{dt} = o, i = 1,2,3$ .The system under investigation has six equilibrium states that can be put in four categories A,B,C,D as follows **:**

A. Fully washed out state.

(i). $\overline{N_1}=0$; $\overline{N_2}=0$; $\overline{N_3}=0$ ……………………… (4.1)

B. States in which some two of three species are washed out and third is not.

(ii). $\overline{N_1}=0$; $\overline{N_2}=0$; $\overline{N_3}=k_3$. ……………………. (4.2)

(iii). $\overline{N_1}=0$; $\overline{N_2}=k_2$; $\overline{N_3}=0$ ……………………… (4.3)

C. Only one of the three species is washed out while the other two are not.

(iv). $\overline{N_1}=0$; $\overline{N_2}=k_2$; $\overline{N_3}=k_3$ …………………. (4.4)

(v). $\overline{N_1}=ck_3-e_1$; $\overline{N_2}=0$; $\overline{N_3}=k_3$ ………………….. (4.5)

This would exists only when $e_1 < ck_3$. When $e_1 = ck_3$ this equilibrium state merges with the equilibrium state **No. (4.2).**

D. The co-existence state or normal steady state

(vi). $\overline{N_1}=\frac{ck_3-(e_1+pk_2)}{1+pq}$; $\overline{N_2}=\frac{k_2+qck_3-qe_1}{1+pq}$; $\overline{N_3}=k_3$ ...…………… (4.6)

This would exists when $e_1 <$ smaller of $\left\{(ck_3 - pk_2) and \left(ck_3 + \frac{k_2}{q}\right)\right\}$

5. THE STABILITY OF THE EQUILIBRIUM STATES:

To this end, we consider slight deviations $U_1(t), U_2(t), U_3(t)$ over the steady state $(\overline{N_1}, \overline{N_2}, \overline{N_3})$:

$N_1=\overline{N_1}+U_1(t)$; $N_2=\overline{N_2}+U_2(t)$; $N_3=\overline{N_3}+U_3(t)$ ………….. (5.1)

where $U_1(t), U_2(t)$ and $U_3(t)$ are small so that their second and higher powers and products can be neglected.

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5.1. FULLY WASHED OUT EQUILIBRIUM STATE: (E_1)

In this case, we get

$$\frac{dU_1}{dt} = -e_1 a_{11} U_1 \quad ; \quad \frac{dU_2}{dt} = k_2 a_{22} U_2 \quad ; \quad \frac{dU_3}{dt} = k_3 a_{33} U_3 \qquad \text{……………. (5.1.1)}$$

the characteristic roots of the system are $-e_1 a_{11},\ k_2 a_{22},\ k_3 a_{33}$. Since two of the three roots are positive, the state is **unstable**.

The equation (5.1.1) yield the solution curves

$$U_1 = U_{10} e^{-e_1 a_{11} t} \quad ; \quad U_2 = U_{20} e^{k_2 a_{22} t} \quad ; \quad U_3 = U_{30} e^{k_3 a_{33} t} \qquad \text{………. (5.1.2)}$$

where U_{10}, U_{20}, U_{30} are initial values of U_1, U_2, U_3 respectively.

Trajectories of Perturbations:

The trajectories (solution curves of (5.1.1)) in the $U_1 - U_2$ plane, $U_2 - U_3$ plane and $U_3 - U_1$ plane are given by

$$\left(\frac{U_1}{U_{10}}\right)^{-k_2 a_{22}} = \left(\frac{U_2}{U_{20}}\right)^{e_1 a_{11}} \; ; \; \left(\frac{U_2}{U_{20}}\right)^{k_3 a_{33}} = \left(\frac{U_3}{U_{30}}\right)^{k_2 a_{22}} \; ; \; \left(\frac{U_3}{U_{30}}\right)^{e_1 a_{11}} = \left(\frac{U_1}{U_{10}}\right)^{-k_3 a_{33}} \qquad \text{………… (5.1.3)}$$

Several different solution curves have observed of which only few of them are discussed in the following figures and the conclusions are presented.

Case: (5.1. (i)) $\quad U_{10} > U_{20} > U_{30}$ **:** $k_2 a_{22} > k_3 a_{33}$

The prey out-number both the predator and the host till the time instants
$t_{12}^* = \frac{1}{(k_2 a_{22} + e_1 a_{11})} \log\left(\frac{U_{10}}{U_{20}}\right)$ and $t_{13}^* = \frac{1}{(k_3 a_{33} + e_1 a_{11})} \log\left(\frac{U_{10}}{U_{30}}\right)$

respectively and there after the prey-commensal declines even though its initial population strength is greater than that of both the predator and the host. Here the predator dominates the host in natural growth rate as well as in its initial population strength.

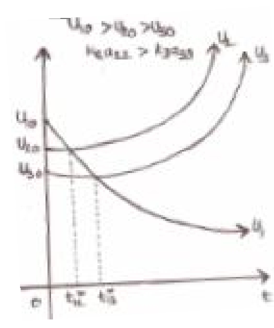

Figure: 1

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Case: (5.1. (ii)) $U_{20} > U_{10} > U_{30}$ **;** $k_2 a_{22} > k_3 a_{33}$

The predator always out-numbers both the prey and the host in natural growth rates as well as in their initial population strengths. In this case the host dominates the prey in its natural growth rate but its initial population strength is less than that of prey. Here the prey out-numbers the host up to the time instant $t_{13}^{*} = \frac{1}{(k_3 a_{33} + e_1 a_{11})} \log\left(\frac{U_{10}}{U_{30}}\right)$ after which the host out- numbers the prey. In this case both the predator and the host are going away from the equilibrium point, while the prey is asymptotic to the equilibrium point.

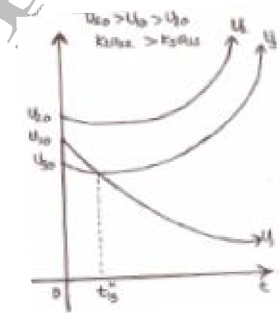

Figure: 2

Case: (5.1. (iii)) $U_{30} > U_{10} > U_{20}$; $k_2 a_{22} > k_3 a_{33}$

In this case the predator dominates both the prey and the host in natural growth rate even though its initial population strength is less than that of both the prey and the host.Here the host out-numbers the predator up to the time instant $t_{23}^{*} = \frac{1}{(k_3 a_{33} - k_2 a_{22})} \log\left(\frac{U_{20}}{U_{30}}\right)$ and there after the predator out-numbers the host. The predator dominates the prey after the time instant $t_{12}^{*} = \frac{1}{(k_2 a_{22} + e_1 a_{11})} \log\left(\frac{U_{10}}{U_{20}}\right)$ after which the dominance is reversed.

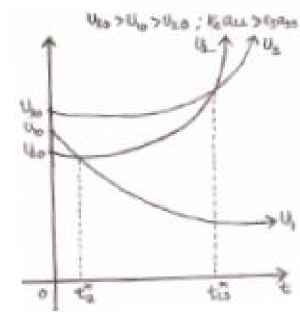

Figure: 3

5.2 PREY-COMMENSAL AND PREDATOR WASHED OUT STATE: (E_2)

In this case, we have

$$\frac{dU_1}{dt} = a_{11}(c - e_1)U_1 \quad ; \quad \frac{dU_2}{dt} = k_2 a_{22} U_2 \quad ; \quad \frac{dU_3}{dt} = -k_3 a_{33} U_3 \qquad \text{……………….. (5.2.1)}$$

Since the characteristic roots of which are $a_{11}(c - e_1), k_2 a_{22}, -k_3 a_{33}$, the state is **unstable.**

The equation (5.2.1) yield the solution curves

$$U_1 = U_{10} e^{a_{11}(c - e_1)t} \quad ; \quad U_2 = U_{20} e^{k_2 a_{22} t} \quad ; \quad U_3 = U_{30} e^{-k_3 a_{33} t} \qquad \text{…………. (5.2.2)}$$

where U_{10}, U_{20}, U_{30} are initial values of U_1, U_2, U_3 respectively.

CASE: A When $c > e_1$ (**i.e. the commensalisms is strong**) then from (5.2.2) we have

$$U_1 = U_{10} e^{a_{11}(c-e_1)t} \quad ; \quad U_2 = U_{20} e^{k_2 a_{22} t} \quad ; \quad U_3 = U_{30} e^{-k_3 a_{33} t} \qquad \text{............ (5.2.3)}$$

<u>Trajectories of Perturbations:</u>

In this case the trajectories of (5.2.2) in the $U_1 - U_2$ plane, $U_2 - U_3$ plane and $U_3 - U_1$ plane are given by

$$\left(\frac{U_1}{U_{10}}\right)^{k_2 a_{22}} = \left(\frac{U_2}{U_{20}}\right)^{(c-e_1)a_{11}} ; \left(\frac{U_2}{U_{20}}\right)^{k_3 a_{33}} = \left(\frac{U_3}{U_{30}}\right)^{-k_2 a_{22}} ; \left(\frac{U_3}{U_{30}}\right)^{(c-e_1)a_{11}} = \left(\frac{U_1}{U_{10}}\right)^{k_3 a_{33}} \qquad \text{........ (5.2.A)}$$

Some solution curves of (5.2.3) are illustrated in the following figures and the conclusions are presented.

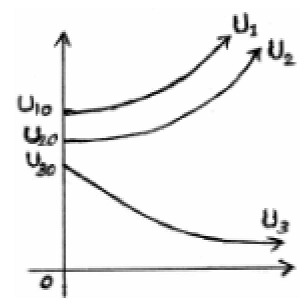

Figure: 4

Case: **(5.2.A (i)**) $U_{10} > U_{20} > U_{30}$: $(c-e_1)a_{11} > k_2 a_{22}$
The prey-commensal always out-numbers both the predator and the host in natural growth rates as well as in their initial population strengths. In this case the prey and the predator species are noted to be going away from the equilibrium point while the host is asymptotic to the equilibrium point.

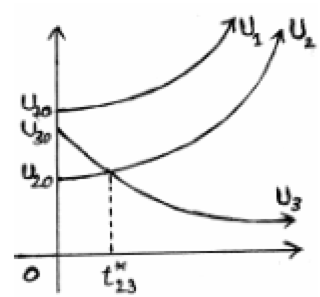

Figure: 5

Case: (5.2.A (ii)) $U_{10} > U_{30} > U_{20}$: $(c-e_1)a_{11} < k_2 a_{22}$

In this case the prey-commensal always out-numbers both the predator and the host in natural growth rates as well as in their initial population strengths.

Here the host out- members the predator till the time instant

$$t_{23}^{\ *} = \frac{1}{(k_2 a_{22} + k_3 a_{33})} \log\left(\frac{U_{30}}{U_{20}}\right)$$ and there after the dominance is reversed.

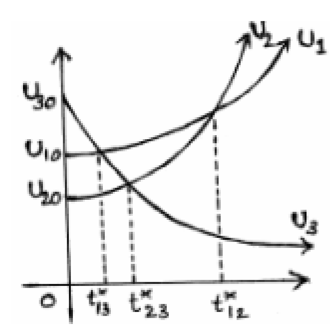

Figure: 6

Case: (5.2.A (iii)) $U_{30} > U_{10} > U_{20}$: $(c-e_1)a_{11} < k_2 a_{22}$

Even though the initial population strengths are less than that of both the prey and the host the predator dominate both the prey and the host in

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natural growth rate. In this case the host and the prey out- number the predator till the time instants $t_{13}^* = \frac{1}{(k_2 a_{22} + (c - e_1) a_{11})} \log\left(\frac{U_{10}}{U_{30}}\right)$ and $t_{23}^* = \frac{1}{(k_2 a_{22} + k_3 a_{33})} \log\left(\frac{U_{30}}{U_{20}}\right)$ respectively after which dominance is reversed. Here the prey out-numbers the predator up to the time instant $t_{12}^* = \frac{1}{(k_2 a_{22} - (c - e_1) a_{11})} \log\left(\frac{U_{10}}{U_{20}}\right)$ after which the dominance is reversed.

CASE: B When $c < e_1$ (**i.e. the commensalisms is weak**) then (5.2.2) becomes

$$U_1 = U_{10} e^{-a_{11}(e_1 - c)t} \quad ; \quad U_2 = U_{20} e^{k_2 a_{22} t} \quad ; \quad U_3 = U_{30} e^{-k_3 a_{33} t} \qquad \ldots\ldots\ldots\ldots\ldots (5.2.4)$$

<u>**Trajectories of Perturbations:**</u>

The trajectories of (5.2.2) in the $U_1 - U_2$ plane, $U_2 - U_3$ plane and $U_3 - U_1$ plane are given by

$$\left(\frac{U_1}{U_{10}}\right)^{k_2 a_{22}} = \left(\frac{U_2}{U_{20}}\right)^{(e_1 - c) a_{11}} ; \left(\frac{U_2}{U_{20}}\right)^{k_3 a_{33}} = \left(\frac{U_3}{U_{30}}\right)^{-k_2 a_{22}} ; \left(\frac{U_3}{U_{30}}\right)^{(e_1 - c) a_{11}} = \left(\frac{U_1}{U_{10}}\right)^{-k_3 a_{33}} \qquad \ldots\ldots\ldots (5.2.B)$$

Some solution curves of (5.2.4) are illustrated in the following figures and the conclusions are presented.

Case (5.2.B (i)): $U_{10} > U_{20} > U_{30}$: $(e_1 - c) a_{11} > k_3 a_{33}$

The prey out- numbers the predator till the time instant $t_{12}^* = \frac{1}{(k_2 a_{22} + (e_1 - c) a_{11})} \log\left(\frac{U_{10}}{U_{20}}\right)$ after which the prey declines. Here the prey out-numbers the host till the time instant $t_{13}^* = \frac{1}{(k_2 a_{22} - (e_1 - c) a_{11})} \log\left(\frac{U_{30}}{U_{10}}\right)$ and there after the host out- numbers the prey. In this case both the prey and the host converge asymptotically to the equilibrium point, while the predator goes far away from the equilibrium point.

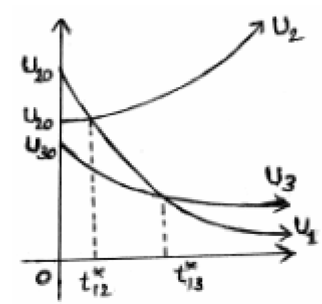

Figure: 7

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Case (5.2.B (ii)): $U_{10} > U_{20} > U_{30}$: $(e_1 - c)a_{11} < k_3 a_{33}$

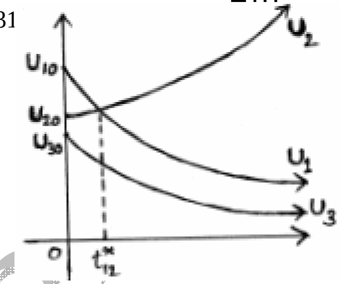

Figure: 8

The prey out-members the predator till the time $t_{12}^{\ *} = \frac{1}{\left(k_2 a_{22} - (c - e_1)a_{11}\right)} \log\left(\frac{U_{10}}{U_{20}}\right)$ and there after the dominance is reversed. In this case both the prey and the host converge asymptotically to the equilibrium point, while the predator goes far away from the equilibrium point.

Case (5.2.B (iii)): $U_{20} > U_{10} > U_{30}$: $(e_1 - c)a_{11} < k_3 a_{33}$

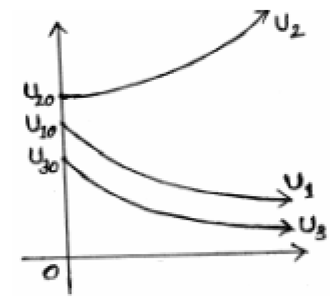

Figure: 9

In this case the predator always out-numbers both the prey and the host in their natural growth rates as well as in their initial population strengths. And the prey out-numbers the host in natural growth rates as well as in its initial population strength. In this case both the prey and the host converge asymptotically to the equilibrium point, while the predator goes far away from the equilibrium point.

CASE: C When $c = e_1$ then (5.2.2) becomes

$$U_1 = U_{10} \quad ; \quad U_2 = U_{20} e^{k_2 a_{22} t} \quad ; \quad U_3 = U_{30} e^{-k_3 a_{33} t} \qquad \text{……….…………. (5.2.5)}$$

Trajectories of Perturbations:

In this case the trajectories of (5.2.2) in the $U_1 - U_2$ plane, $U_2 - U_3$ plane and $U_3 - U_1$ plane are given by

$$\frac{U_1}{U_{10}} = 1 \quad ; \quad \left(\frac{U_2}{U_{20}}\right)^{k_3 a_{33}} = \left(\frac{U_3}{U_{30}}\right)^{-k_2 a_{22}} \qquad \text{…..……….………. (5.2.C)}$$

A few solution curves of (5.2.3) are illustrated in the following figures and the conclusions are presented.

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Case: (5.2.C (i)) $U_{10} > U_{20} > U_{30}$

In this case the host converges asymptotically to equilibrium point while the prey and the predator go away from the equilibrium point. Here the prey out -numbers the predator till the time $t_{12}^{\,*} = \frac{1}{k_2 a_{22}} \log\left(\frac{U_{10}}{U_{20}}\right)$ and there after the predator out-numbers the prey.

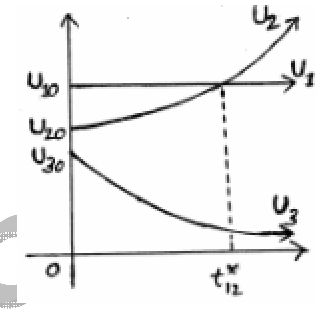

Figure: 10

Case: (5.2.C (ii)) $U_{10} > U_{30} > U_{20}$

The predator dominates both the prey and the host in their natural growth rates but its initial population strength is less than that of both the prey and the host. Here both the prey and the host out- numbers the predator till the time instants $t_{12}^{\,*} = \frac{1}{k_2 a_{22}} \log\left(\frac{U_{10}}{U_{20}}\right)$ and $t_{23}^{\,*} = \frac{1}{(k_2 a_{22} + k_3 a_{33})} \log\left(\frac{U_{30}}{U_{20}}\right)$ respectively and then the dominance is reversed.

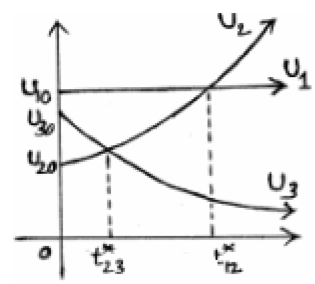

Figure: 11

Case: (5.2.C (iii)) $U_{20} > U_{10} > U_{30}$

The predator always out-numbers both the prey and the host in natural growth rates as well as in their initial population strengths. In this case the prey has a constant growth rate and the predator goes far away from the equilibrium point while the host is asymptotic to equilibrium point .

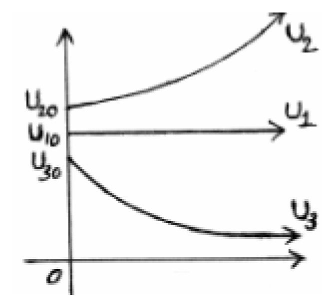

Figure: 12

5.3 PREY-COMMENSAL AND HOST WASHED OUT EQUILIBRIUM STATE: (E_3)

Here we have

$$\frac{dU_1}{dt} = -a_{11}(e_1 + p)U_1 \quad ; \quad \frac{dU_2}{dt} = a_{22} q k_2 U_1 - k_2 a_{22} U_2 \quad ; \quad \frac{dU_3}{dt} = k_3 a_{33} U_3 \qquad \text{………………..} \; (5.3.1)$$

the characteristic roots of which are $-a_{11}(e_1 + p), -k_2 a_{22}, k_3 a_{33}$. Hence the state is **unstable.**

The equation (5.3.1) yield the solution curves

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$$U_1 = U_{10} e^{-a_{11}(e_1+p)t} \quad ; \quad U_2 = \frac{a_{22} q k_2 U_{10}}{[k_2 a_{22} - a_{11}(e_1+p)]} e^{-a_{11}(e_1+p)t} + \left[U_{20} - \frac{a_{22} q k_2 U_{10}}{[k_2 a_{22} - a_{11}(e_1+p)]} \right] e^{-k_2 a_{22} t}$$

$$U_3 = U_{30} e^{k_3 a_{33} t} \qquad \text{.....................} (5.3.2)$$

When $U_{20} = \frac{a_{22} q k_2 U_{10}}{[k_2 a_{22} - a_{11}(e_1+p)]}$ then (5.3.2) becomes

$$U_1 = U_{10} e^{-a_{11}(e_1+p)t} \quad ; \quad U_2 = U_{20} e^{-a_{11}(e_1+p)t} \quad ; \quad U_3 = U_{30} e^{k_3 a_{33} t} \qquad \text{....................} (5.3.3)$$

Trajectories of Perturbations:

In this case the trajectories of (5.3.3) in the $U_1 - U_2$ plane, $U_2 - U_3$ plane and $U_3 - U_1$ plane is given by

$$\frac{U_1}{U_{10}} = \frac{U_2}{U_{20}} \quad ; \quad \left(\frac{U_2}{U_{20}}\right)^{-k_3 a_{33}} = \left(\frac{U_3}{U_{30}}\right)^{a_{11}(e_1+p)} \quad ; \quad \left(\frac{U_3}{U_{30}}\right)^{(e_1+p)a_{11}} = \left(\frac{U_1}{U_{10}}\right)^{-k_3 a_{33}} \qquad \text{.......} (5.3.A)$$

Some solution curves of (5.3.3) are illustrated in the figures with some remarks.

Case: (5.3.A (i)) $U_{10} > U_{20} > U_{30}$

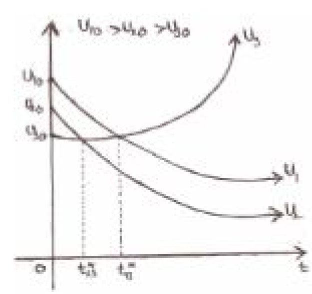


Figure: 13

The host dominates both the prey and predator in their natural growth rates, but itsinitial population strength is less than that of both the prey and predator. The prey and the predator out- numbers the host till the time instants

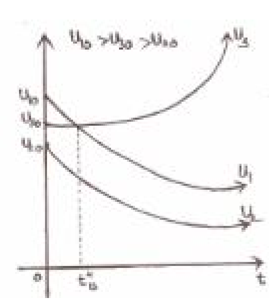
$$t_{13}^* = \frac{1}{(k_3 a_{33} + (e_1+p) a_{11})} \log\left(\frac{U_{10}}{U_{30}}\right)$$

and $t_{23}^* = \frac{1}{(k_3 a_{33} + (e_1+p) a_{11})} \log\left(\frac{U_{20}}{U_{30}}\right)$ respectively after which both the prey and the predator declines

.

Case: (5.3.A (ii)) $U_{10} > U_{30} > U_{20}$

The host dominates both the prey and predator in its natural growth rates. The prey out- numbers the host till the time instant



$$t_{13}^* = \frac{1}{(k_3 a_{33} + (e_1+p) a_{11})} \log\left(\frac{U_{10}}{U_{30}}\right)$$ after which the dominance is reversed.

Figure: 14

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Case: (5.3.A (iii)) $U_{30} > U_{10} > U_{20}$

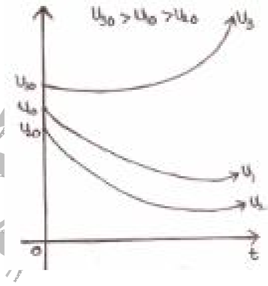

Figure: 15

The host always out-numbers both the prey and the predator in natural growth rates as well as in their initial population strengths. In this case both the prey and the predator converge asymptotically to the equilibrium point while the host goes away from the equilibrium point.

5.4 PREY-COMMENSAL WASHED OUT EQUILIBRIUMSTATE: (E_4)

Here we have

$$\frac{dU_1}{dt} = a_{11}[ck_3 - (pk_2 + e_1)]U_1 \quad ; \quad \frac{dU_2}{dt} = a_{22}qk_2U_1 - k_2a_{22}U_2 \quad ; \quad \frac{dU_3}{dt} = -k_3a_{33}U_3 \qquad \text{……….. (5.4.1)}$$

And the characteristic roots of which are $a_{11}[ck_3 - (pk_2 + e_1)], -k_2a_{22}, -k_3a_{33}$.

Case: A When $e_1 > ck_3 - pk_2$, all the three roots are negative and hence the state is **stable**.

In this case the solutions of (5.4.1) are

$$U_1 = U_{10}e^{-a_{11}[e_1 - (ck_3 - pk_2)]t}$$

$$U_2 = \frac{a_{22}qk_2U_{10}}{[k_2a_{22} - a_{11}[e_1 - (ck_3 - pk_2)]]}e^{-a_{11}[e_1 - (ck_3 - pk_2)]t} + \left[U_{20} - \frac{a_{22}qk_2U_{10}}{[k_2a_{22} - a_{11}[e_1 - (ck_3 - pk_2)]]}\right]e^{-k_2a_{22}t}$$

$$U_3 = U_{30}e^{-k_3a_{33}t} \qquad \text{……. ………… (5.4.2)}$$

when $e_1 > ck_3 - pk_2$ and $U_{20} = \dfrac{a_{22}qk_2U_{10}}{[k_2a_{22} - a_{11}[e_1 - (ck_3 - pk_2)]]}$ then (5.4.2) becomes

$$U_1 = U_{10}e^{-a_{11}[e_1 - (ck_3 - pk_2)]t} \quad ; \quad U_2 = U_{20}e^{-a_{11}[e_1 - (ck_3 - pk_2)]t} \quad ; \quad U_3 = U_{30}e^{-k_3a_{33}t} \qquad \text{………… (5.4.3)}$$

Trajectories of Perturbations:

In this case the trajectories of (5.4.3) in the $U_1 - U_2$ plane, $U_2 - U_3$ plane and $U_3 - U_1$ plane are given by

$$\frac{U_1}{U_{10}} = \frac{U_2}{U_{20}}; \left(\frac{U_2}{U_{20}}\right)^{k_3a_{33}} = \left(\frac{U_3}{U_{30}}\right)^{a_{11}[e_1 - (ck_3 - pk_2)]} \quad ; \quad \left(\frac{U_3}{U_{30}}\right)^{a_{11}[e_1 - (ck_3 - pk_2)]} = \left(\frac{U_1}{U_{10}}\right)^{k_3a_{33}} \qquad \text{……………. (5.4.A)}$$

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Several solution curves of (5.4.3) have observed of which only few of them are illustrated here under.

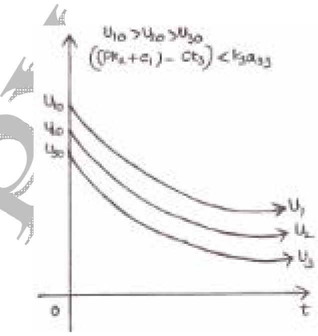

Figure: 16

Case: (5.4.A (i)) $U_{10} > U_{20} > U_{30}$; $[(pk_2 + e_1) - ck_3] < k_3 a_{33}$

In this case the prey always out-numbers both the predator and host in natural growth rates as well as in their initial populations strengths. Here the predator always out-numbers the host. However the three species converge asymptotically to the equilibrium point. Hence the state is **stable.**

Case: (5.4.A (ii)) $U_{10} > U_{20} > U_{30}$; $[(pk_2 + e_1) - ck_3] > k_3 a_{33}$

The host dominates both the prey and the predator in natural growth rates but its initial population strength is less than that of both the prey and the predator.

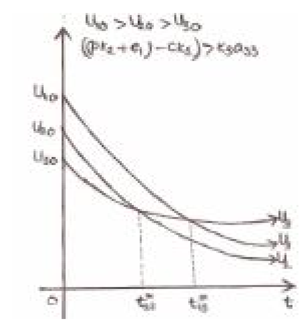

Figure: 17

Initially both the prey and the predator out- numbers the host till the time instants $t_{13}{}^{*} = \frac{1}{[[ck_3 - (pk_2 + e_1)] - k_3 a_{33}]} \log\left(\frac{U_{10}}{U_{30}}\right)$ and

$$t_{23}{}^{*} = \frac{1}{[[ck_3 - (pk_2 + e_1)] - k_3 a_{33}]} \log\left(\frac{U_{20}}{U_{30}}\right)$$

respectively after which the host out-numbers both the prey and the predator. Further all the three species converge asymptotically to the equilibrium point.

Case: B When $e_1 < ck_3 - pk_2$, one of the three roots is positive so that the state is **unstable**.

In this case the solution of (5.4.1) is

$$U_1 = U_{10} e^{a_{11}[ck_3 - (pk_2 + e_1)]t}$$

$$U_2 = \frac{a_{22} q k_2 U_{10}}{[k_2 a_{22} + a_{11}[ck_3 - (pk_2 + e_1)]]} e^{a_{11}[ck_3 - (pk_2 + e_1)]t} + \left[U_{20} - \frac{a_{22} q k_2 U_{10}}{[k_2 a_{22} + a_{11}[ck_3 - (pk_2 + e_1)]]}\right] e^{-k_2 a_{22} t}$$

$$U_3 = U_{30} e^{-k_3 a_{33} t} \qquad \text{………..…………. (5.4.4)}$$

When $e_1 < ck_3 - pk_2$ and $U_{20} = \frac{a_{22} q k_2 U_{10}}{[k_2 a_{22} + a_{11}[ck_3 - (pk_2 + e_1)]]}$ then (5.4.4) becomes

$$U_1 = U_{10} e^{a_{11}[ck_3 - (pk_2 + e_1)]t} \; ; \; U_2 = U_{20} e^{a_{11}[ck_3 - (pk_2 + e_1)]t} \; ; \; U_3 = U_{30} e^{-k_3 a_{33} t} \qquad \text{………………. (5.4.5)}$$

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Trajectories of Perturbations:

In this case the trajectories of (5.4.5) in the $U_1 - U_2$ plane, $U_2 - U_3$ plane and $U_3 - U_1$ plane are given by

$$\frac{U_1}{U_{10}} = \frac{U_2}{U_{20}} \;;\; \left(\frac{U_2}{U_{20}}\right)^{-k_3 a_{33}} = \left(\frac{U_3}{U_{30}}\right)^{[ck_3 - (pk_2 + e_1)]} \;;\; \left(\frac{U_3}{U_{30}}\right)^{[ck_3 - (pk_2 + e_1)]} = \left(\frac{U_1}{U_{10}}\right)^{-k_3 a_{33}} \quad \text{………………… (5.4.B)}$$

Some solution curves of (5.4.5) are illustrated in the following figures and the conclusions are presented.

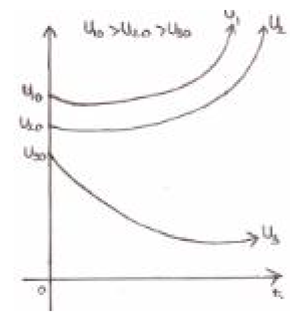

Figure: 18

Case: (5.4.B (i)) $U_{10} > U_{20} > U_{30}$

In this case the prey always out-numbers both the predator and host and the predator always out-numbers the host in natural growth rates as well as in population strengths. Further the prey and the predator go far away from the equilibrium point while the host is asymptotic to the equilibrium point.

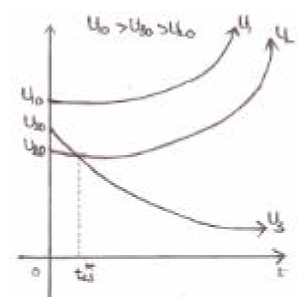

Figure: 19

Case: (5.4.B (ii)) $U_{10} > U_{30} > U_{20}$

In this case both the prey and predator go away from the equilibrium point while the host is asymptotic to the equilibrium point.

Case: C When $e_1 = ck_3 - pk_2$, one of the three roots would be zero so that the state is **unstable.**

In this case the solutions of (5.4.1) are

$$U_1 = U_{10} \;;\; U_2 = U_{10}q + [U_{20} - U_{10}q]e^{-k_2 a_{22} t} \;;\; U_3 = U_{30}e^{-k_3 a_{33} t} \quad \text{….. …………. (5.4.6)}$$

when $U_{20} = qU_{10}$ then (5.4.6) becomes

$$U_1 = U_{10} \quad ; \quad U_2 = U_{20} \quad ; \quad U_3 = U_{30}e^{-k_3 a_{33} t} \quad \text{………………. (5.4.7)}$$

Trajectories of Perturbations:

The trajectories of (5.4.7) in the $U_1 - U_2$ plane, $U_2 - U_3$ plane and $U_3 - U_1$ plane are given by

$$\frac{U_1}{U_{10}} = 1 \quad ; \quad \frac{U_2}{U_{20}} = 1 \quad \text{……………… (5.4.C)}$$

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Some solution curves of (5.4.7) are illustrated in the following figures and the conclusions are presented.

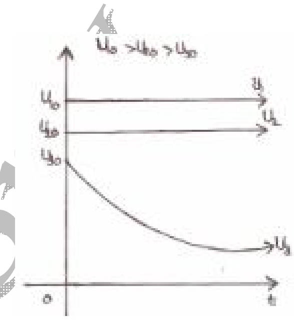
Figure: 20

Case: (5.4.C (i)) $U_{10} > U_{20} > U_{30}$

In this case both the prey and the predator have a constant growth rate in nature and the host is asymptotic to the equilibrium point

Case: (5.4.C (ii)) $U_{30} > U_{10} > U_{20}$

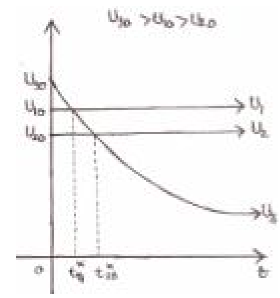
Figure: 21

Here the host out- numbers both the prey and the predator till the time instants

$t_{13}{}^{*} = \frac{1}{k_3 a_{33}} \log\left(\frac{U_{30}}{U_{10}}\right)$ and $t_{23}{}^{*} = \frac{1}{k_3 a_{33}} \log\left(\frac{U_{30}}{U_{20}}\right)$ respectively after which the host declines further.

5.5 PREDATOR WASHED OUT EQUILIBRIUM STATE: (E_5)

Here we have

$$\frac{dU_1}{dt} = a_{11}(e_1 - ck_3)U_1 - a_{11}p(ck_3 - e_1)U_2 - a_{11}(ck_3 - e_1)U_3 \ ; \quad \frac{dU_2}{dt} = a_{22}[k_2 + qck_3 - qe_1]U_2 \ ;$$

$$\frac{dU_3}{dt} = -k_3 a_{33} U_3 \qquad \text{.................. (5.5.1)}$$

The characteristic equation of which is $(\lambda - a_{11}(e_1 - ck_3))\,(\lambda - a_{22}(ek_2 + qck_3 - qe_1))\,(\lambda + k_3 a_{33}) = 0$ and whose roots are $a_{11}(e_1 - ck_3), a_{22}(k_2 + qck_3 - qe_1), -k_3 a_{33}$.

Case: A When e_1 does not lies between $ck_3 + \frac{k_2}{q}$ and ck_3, all the three roots are negative, hence the state is **stable**.

In this case the solution curves of (5.5.1) are as follows

$$U_1 = \alpha_1 e^{-\gamma_1 t} - \alpha_2 e^{-k_3 a_{33} t} + [U_{10} - (\alpha_1 - \alpha_2)]e^{-\gamma_2 t};\ U_2 = U_{20} e^{-\gamma_1 t};\ U_3 = U_{30} e^{-k_3 a_{33} t} \quad \text{......... (5.5.2)}$$

Where $\alpha_1 = \frac{a_{11} p (e_1 - ck_3) U_{20}}{a_{22}(k_2 + qck_3 - qe_1) - a_{11}(e_1 - ck_3)}$; $\alpha_2 = ca_{11}(ck_3 - e_1)U_{30}$;

$\gamma_1 = a_{22}(qe_1 - (k_2 + qck_3))$; $\gamma_2 = a_{11}(ck_3 - e_1)$

When e_1 does not lies between $ck_3 + \frac{k_2}{q}$ and ck_3 and $U_{10} = \alpha_1 - \alpha_2$ then (5.5.2) becomes

$$U_1 = U_{10} e^{-\gamma_1 t} + \alpha_2 \left[e^{-\gamma_1 t} - e^{-k_3 a_{33} t}\right] \ ;\ U_2 = U_{20} e^{-\gamma_1 t} \ ;\ U_3 = U_{30} e^{-k_3 a_{33} t} \quad \text{......... (5.5.3)}$$

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Trajectories of Perturbations:

In this case the trajectories of (5.4.3) in the $U_1 - U_2$ plane, $U_2 - U_3$ plane and $U_3 - U_1$ plane are respectively given by

x = (1+M) y-Mz ; $\left(\frac{U_2}{U_{20}}\right)^{k_3 a_{33}} = \left(\frac{U_3}{U_{30}}\right)^{\gamma_1}$ ……….....…………… (5.5.A)

where $x = \frac{U_1}{U_{10}}$; $y = \frac{U_2}{U_{20}}$; $z = \frac{U_3}{U_{30}}$ and $M = \frac{\alpha_2}{U_{10}}$

Some solution curves of (5.5.3) are illustrated in the following figures and the conclusions are presented.

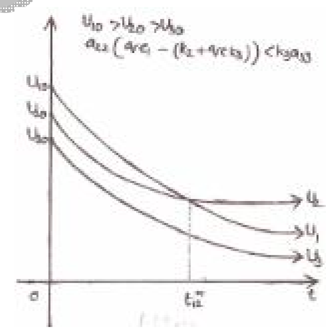

Figure: 22

Case: (5.5.A (i)) $U_{10} > U_{20} > U_{30}$; $\gamma_1 < k_3 a_{33}$

In this case prey out-numbers the predator till the time instant

$t_{12}^{*} = \frac{1}{\gamma_1 - k_3 a_{33}} \log\left(\frac{U_{10} - U_{20} + \alpha_2}{\alpha_2}\right)$ after which the dominance is reversed.

Here the prey always out-numbers the host in natural growth rate. Here all the three species converge asymptotically to the equilibrium point.

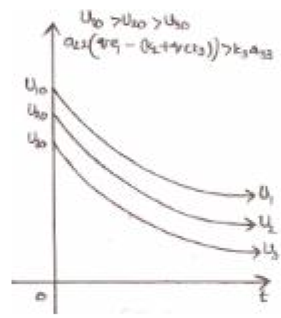

Figure: 23

Case: (5.5.A (ii)) $U_{10} > U_{20} > U_{30}$; $\gamma_1 > k_3 a_{33}$

Here the prey always out-number both the predator and the host , the predator always out-numbers the host in natural growth rates as well as in its initial population strengths. However the three species converge asymptotically to the equilibrium point.

Case: B When $e_1 < ck_3 + \frac{k_2}{q}$, one of three roots is positive so that the state is **unstable**.

In this case the solution curves of (5.5.1) are as follows

$$U_1 = \alpha_1 e^{\gamma_1 t} - \alpha_2 e^{-k_3 a_{33} t} + \left[U_{10} - (\alpha_1 - \alpha_2)\right] e^{-\gamma_2 t} ;\ U_2 = U_{20} e^{\gamma_1 t} ; U_3 = U_{30} e^{-k_3 a_{33} t} \quad \text{.......... (5.5.4)}$$

where $\alpha_1 = \frac{a_{11} p (e_1 - ck_3) U_{20}}{a_{22}(k_2 + qck_3 - qe_1) - a_{11}(e_1 - ck_3)}$; $\alpha_2 = ca_{11}(ck_3 - e_1) U_{30}$;

$\gamma_1 = a_{22}(qe_1 - (k_2 + qck_3))$; $\gamma_2 = a_{11}(ck_3 - e_1)$

When $e_1 < ck_3 + \frac{k_2}{q}$ and $U_{10} = \alpha_1 - \alpha_2$ then (5.5.4) becomes

$$U_1 = U_{10} e^{\gamma_1 t} + \alpha_2 \left[e^{\gamma_1 t} - e^{-k_3 a_{33} t}\right] ;\ U_2 = U_{20} e^{\gamma_1 t} ;\ U_3 = U_{30} e^{-k_3 a_{33} t} \quad \text{.......... (5.5.5)}$$

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Trajectories of Perturbations:

In this case the trajectories of (5.5.5) in the $U_1 - U_2$ plane, $U_2 - U_3$ plane and $U_3 - U_1$ plane are respectively given by

$$x = (1+M)\, y - Mz \quad ; \quad \left(\frac{U_2}{U_{20}}\right)^{-k_3 a_{33}} = \left(\frac{U_3}{U_{30}}\right)^{\gamma_1} \qquad \text{……….……………. (5.5.B)}$$

where $x = \frac{U_1}{U_{10}}$; $y = \frac{U_2}{U_{20}}$; $z = \frac{U_3}{U_{30}}$ and $M = \frac{\alpha_2}{U_{10}}$

Some solution curves of (5.5.5) are illustrated in the following figures with some remarks.

Case: (5.5.B (i)) $U_{10} > U_{20} > U_{30}$

Here the prey always out-numbers both the predator and the host and the predator always out-numbers the host in natural growth rates as well as in its initial population strengths. In this case the host species is asymptotic to the equilibrium point while other two species go far away from the equilibrium point.

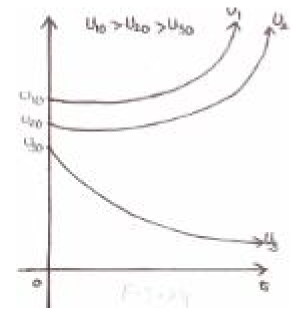

Figure: 24

Case: (5.5.B (ii)) $U_{10} > U_{30} > U_{20}$

The prey dominates both the predator and host in natural growth rates as well as in their initial population strengths. In this case the host out-numbers the predator up to the time instant $t_{23}^{*} = \frac{1}{\gamma_1 + k_3 a_{33}} \log\left(\frac{U_{30}}{\alpha_{20}}\right)$ after which the dominance is reversed.

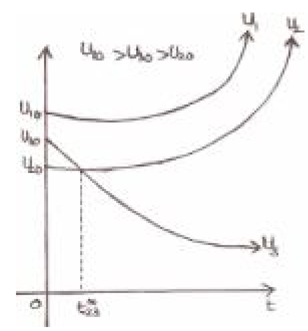

Figure: 25

Case: C When $e_1 = ck_3 + \frac{k_2}{q}$, one of three roots would be zero so that the state is **unstable**.

In this case the equations (5.5.1) yield the solution curves

$$U_1 = \beta_1^{*} - \beta_2^{*} e^{-k_3 a_{33} t} + \left[U_{10} - \left(\beta_1^{*} - \beta_2^{*}\right)\right] e^{a_{11}(e_1 - ck_3)t} ; \; U_2 = U_{20} ; U_3 = U_{30} e^{-k_3 a_{33} t} \qquad \text{….……. (5.5.6)}$$

where $\beta_1^{*} = \frac{a_{11} p (ck_3 - e_1) U_{20}}{a_{11}(e_1 - ck_3)}$ $\beta_2^{*} = \frac{c a_{11}(ck_3 - e_1) U_{30}}{k_3 a_{33} + a_{11}(e_1 - ck_3)}$

When $e_1 = ck_3 + \frac{k_2}{q}$ and $U_{10} = \beta_1^{*} - \beta_2^{*}$ then (5.5.6) becomes

$$U_1 = \left(U_{10} + \beta_2^{*}\right) - \beta_2^{*} e^{-k_3 a_{33} t} \; ; \; U_2 = U_{20} \; ; \; U_3 = U_{30} e^{-k_3 a_{33} t} \qquad \text{…...……. (5.5.7)}$$

Trajectories of Perturbations:

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In this case the trajectories of (5.5.7) in the $U_1 - U_2$ plane, $U_2 - U_3$ plane and $U_3 - U_1$ plane are given by

$$\frac{U_2}{U_{20}} = 1 \ ; \ x = 1 + (1-z)M \qquad \text{......................} \ (5.4.C)$$

Where $x = \dfrac{U_1}{U_{10}}$; $z = \dfrac{U_3}{U_{30}}$ and $M = \dfrac{\beta_2}{U_{10}}$

Some solution curves of (5.5.7) are illustrated in the following figures and the conclusions are presented.

Case: (5.5.C (i)) $U_{10} > U_{20} > U_{30}$

The prey dominates the predator till the time instant $t_{12}{}^{*} = \dfrac{1}{k_3 a_{33}} \log\left(\dfrac{U_{10} - U_{20} + \beta_2{}^{*}}{\beta_2{}^{*}}\right)$ after which it declines while the predator has a constant growth rate in nature. Further both the host and the prey converge asymptotically to the equilibrium point.

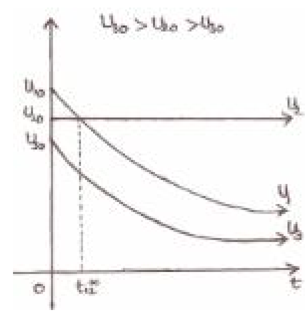

Figure: 26

5.6 CO-EXISTING STATE: (E_6)

In this case, we have

$$\frac{dU_1}{dt} = -\frac{a_{11}[ck_3 - (e_1 + pk_2)]}{1+pq}U_1 - \frac{a_{11}p[ck_3 - (e_1 + pk_2)]}{1+pq}U_2 + \frac{ca_{11}[ck_3 - (e_1 + pk_2)]}{1+pq}U_3 \ ;$$

$$\frac{dU_2}{dt} = -\frac{a_{22}[k_2 + qck_3 - qe_1]}{1+pq}U_1 - \frac{a_{22}[k_2 + qck_3 - qe_1]}{1+pq}U_2 \ ; \quad \frac{dU_3}{dt} = -k_3 a_{33} U_3 \qquad \text{.................} \ (5.6.1)$$

The characteristic equation is $\left(\lambda^2 + (\alpha+\beta)\lambda + (1+pq)\alpha\beta\right)(\lambda + k_3 a_{33}) = 0$ and whose roots are $(\lambda_1, \lambda_2, \lambda_3) = \dfrac{-(\alpha+\beta) \pm \sqrt{(\alpha+\beta)^2 - 4\alpha\beta(1+pq)}}{2}, \ -k_3 a_{33}$. These roots are real or complex according as $(\alpha+\beta)^2 \geq 4\alpha\beta(1+pq)$ or $(\alpha+\beta)^2 < 4\alpha\beta(1+pq)$.In any case, the

roots are negative when they are real or has negative real part when they are complex.

Hence the system is always **stable**.

The equations (5.6.1) yield the solutions

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$$U_1(t)=\frac{(\lambda_1+\beta)[U_{10}(\lambda_1+k_3a_{33})+c\alpha U_{30}]-\alpha pU_{20}(\lambda_1+k_3a_{33})}{(\lambda_1-\lambda_2)(\lambda_1+k_3a_{33})}e^{\lambda_1 t}$$
$$+\frac{(\lambda_2+\beta)[U_{10}(\lambda_1+k_3a_{33})+c\alpha U_{30}]-\alpha pU_{20}(\lambda_2+k_3a_{33})}{(\lambda_2-\lambda_1)(\lambda_2+k_3a_{33})}e^{\lambda_2 t}+\frac{(\beta-k_3a_{33})c\alpha U_{30}}{(\lambda_1+k_3a_{33})(\lambda_2+k_3a_{33})}e^{-k_3a_{33}t}$$
$$U_2(t)=\frac{U_{20}(\lambda_1+\alpha)(\lambda_1+k_3a_{33})+\beta qU_{10}(\lambda_1+k_3a_{33})+c\alpha\beta qU_{30}}{(\lambda_1-\lambda_2)(\lambda_1+k_3a_{33})}e^{\lambda_1 t}$$
$$+\frac{U_{20}(\lambda_2+\alpha)(\lambda_2+k_3a_{33})+\beta qU_{10}(\lambda_2+k_3a_{33})+c\alpha\beta qU_{30}}{(\lambda_2-\lambda_1)(\lambda_2+k_3a_{33})}e^{\lambda_2 t}+\frac{c\alpha\beta qU_{30}}{(\lambda_1+k_3a_{33})(\lambda_2+k_3a_{33})}e^{-k_3a_{33}t}$$
$$U_3=U_{30}e^{-k_3a_{33}t} \qquad \text{……………………………….. (5.6.2)}$$

<u>Case A:</u> When $(\alpha+\beta)^2 \geq 4\alpha\beta(1+pq)$ then all the three roots are negative real and hence the equilibrium state is **stable**.

Some solution curves of (5.6.2) are illustrated here under passing some remarks.

Case: (5.6.A (i)) $U_{10} > U_{20} > U_{30}$; $\lambda_1 > \lambda_2 > \lambda_3$

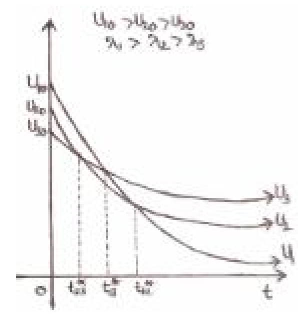

Figure: 27

In this case commensal out-number both the predator and the host for some time after which the dominance is reversed. Here the predator out-numbers the host for some time after which the host out-numbers the predator. However the three species converge asymptotically to the equilibrium point.

Case: (5.6.A (ii)) $U_{10} > U_{20} > U_{30}$; $\lambda_1 > \lambda_3 > \lambda_2$

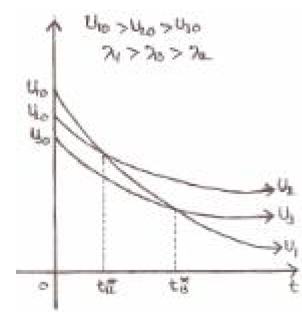

Figure: 28

The prey out-number both the predator and the host for some time after which both the predator and the host out-number the prey-commensal. In this case the three species converge asymptotically to the equilibrium point.

Case: (5.6.A (iv)) $U_{10} > U_{20} > U_{30}$; $\lambda_3 > \lambda_2 > \lambda_1$

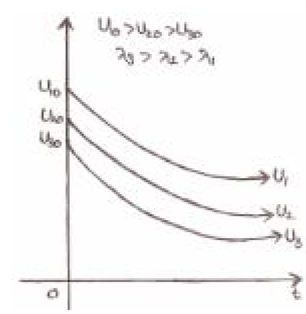

Figure: 29

In this case the prey always out-numbers both the predator and the host and thepredator out-numbers the host in natural growth rates as well as in its initial population strengths. However the three species converges asymptotically to the equilibrium point.

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Case B: When $(\alpha+\beta)^2 < 4\alpha\beta(1+pq)$ then the two roots (λ_1,λ_2) of three are complex with negative real part and the third is (λ_3) negative real and hence the equilibrium state is **stable**.

In this case some solution curves of (5.6.2) are illustrated in the following figures and the conclusions are presented.

Case: (5.6.B (i)) $U_{10} > U_{20} > U_{30}$; $|\lambda_1| > |\lambda_2| > |\lambda_3|$

In this case the prey dominates both the predator and the host in their initial population strengths. Here all the three species converge asymptotically to the equilibrium point.

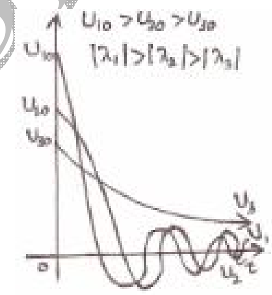

Figure: 30

Case: (5.6.B (ii)) $U_{20} > U_{10} > U_{30}$; $|\lambda_1| > |\lambda_3| > |\lambda_2|$

The predator dominates both the prey and the host and the prey also dominates the host in their initial population strengths. Here the three species converge asymptotically to the equilibrium point.

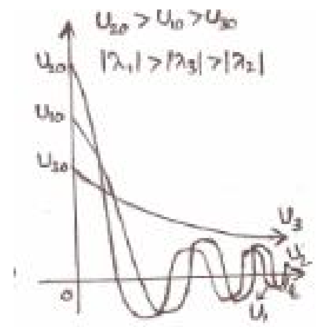

Figure: 31

Case: (5.6.B (iii)) $U_{30} > U_{20} > U_{10}$; $|\lambda_3| > |\lambda_1| > |\lambda_2|$

In this case the host always out-numbers both the prey and the predator in natural growth rates as well as in their initial population strengths. However thethree species converge asymptotically to equilibrium point.

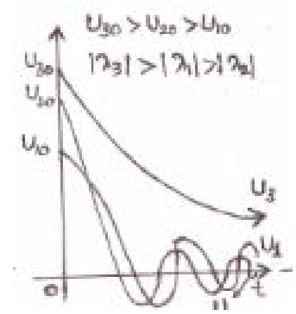

Figure: 32

THRESHOLD DIAGRAM:

Summing up the above results, the threshold results are illustrated by the following threshold diagram figure 33.

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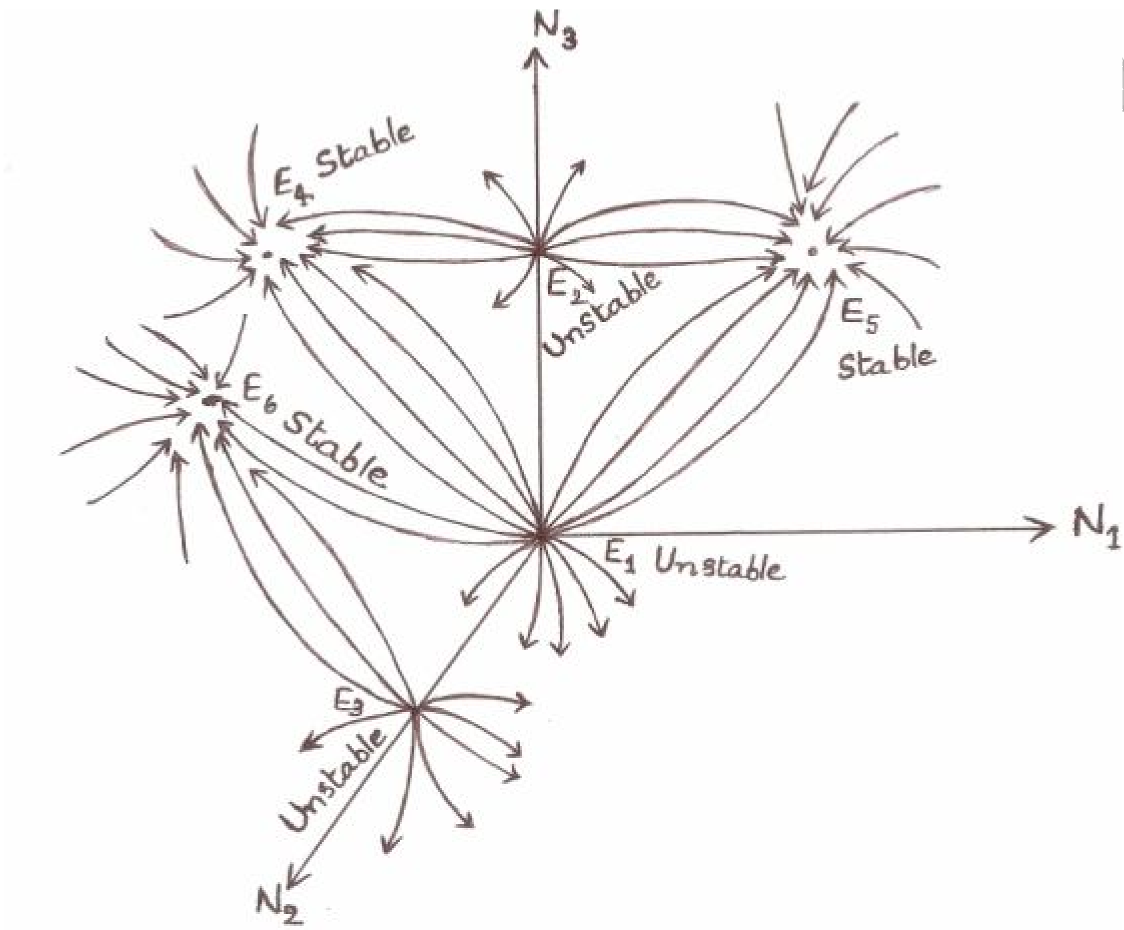

Figure: 33

Threshold diagram in the positive octant $N_1 \geq 0, N_2 \geq 0, N_3 \geq 0$

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