

**A STUDY ON THE GLOBAL STABILITY OF AN AMMENSAL
SPECIES WITH UNLIMITED RESOURCES – THE ENEMY SPECIES
WITH LIMITED RESOURCES**

K.V.L.N.Acharyulu¹ & N. Ch. Pattabhi Ramacharyulu²

¹Faculty in Mathematics, Department of Mathematics
Bapatla Engineering College, Bapatla, India.

Email: kvlna@yahoo.com

²Professor (Retd.) of Mathematics, Department of Mathematics & Humanities
National Institute of Technology, Warangal, India.

ABSTRACT

In this paper the global stability of a Mathematical model of an Ammensal species with unlimited resources and the enemy species with limited resources is examined by Liapunov's stability criteria. It is extracted by constructing a suitable Liapunov's function for evaluating the global stability of the model at fully washed out state where the mortality rates of both the species are greater than their birth rates respectively.

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Key words: Equilibrium states, Stability, Liapunov's function for global stability

1) Introduction:

K.V.L.N.Acharyulu and N.ch.Pattabhi Ramacharyulu studied the Local stability of an Ammensal- enemy eco-system on the quasi-linear basic balancing equations. Local stability analysis for an Ammensal- enemy eco-system with various resources in different cases has been also carried out in the author's earlier work [2,3,5,7]. The present study is concentrated on the establishment of the global stability at fully washed out state where the mortality rates of both the species are greater than their birth rates by employing a property constructed by Liapunov's function with Liapunov's criteria for global stability

Brief History of Liapunov's Stability Analysis:

Liapunov's Analysis yields stability information directly without solving the differential equations involved in the system. Hence it is also called Liapunov's direct method to detect the criteria for global stability. It is based on the chief characteristic of constructing a scalar function called Liapunov's function. A.M. Liapunov initiated a meritable method in 1892 to probe the global stability of equilibrium points in the case of linear and non-linear systems. The stability behaviour of solutions of linear and weakly non-linear system is done by using the techniques of variation of constants formulae and integral inequalities. So this analysis is

bounded to a small neighborhood of operating point i.e., local stability. Further, the techniques used therein require explicit knowledge of solutions of corresponding linear systems. Hence, the stability behaviour of a physical system is curbed by these limitations. In this connection Several autors like Lotka[15], Kapur[13],Pattabhi Ramacharyulu[1-11],Lakiminarayan[14] and Bhaskararama Sarma[12] etc. applied this method in various situations for global stability.

2) An outline sketch of Liapunov’s Method for Global stability:

Consider an autonomous system

$$\frac{dx}{dt} = F_1(x, y) \text{ and } \frac{dy}{dt} = F_2(x, y) \tag{1}$$

Assume that this system has an isolated initial point taken as (0, 0). Consider a function E(x,y) possessing continuous partial derivatives along with the path of (1).This path is represented by C= [(x (t), y (t)] in the parametric form. E(x,y) can be regarded as a function of ‘t’ along C with rate of change

If the total energy of physical system has a local minimum at a certain equilibrium point then the point is said to be stable .Liapunov generalized this principle by constructing a function E(N₁, N₂) whose rate of change is given by

$$\frac{\partial E}{\partial t} = \frac{\partial E}{\partial N_1} \cdot \frac{\partial N_1}{\partial t} + \frac{\partial E}{\partial N_2} \cdot \frac{\partial N_2}{\partial t} = \frac{\partial E}{\partial N_1} F_1 + \frac{\partial E}{\partial N_2} F_2 \tag{2}$$

corresponding to the system.

(ii)Theorem (A): If there exists a Liapunov’s function E (x,y) for the system (1), then the critical point (0,0) is stable. Further, if this function has additional property that the function (2) is negative definite, then the critical point (0, 0) is asymptotically stable. (3)

The following theorem provides to ascertain the definiteness of a Liapunov’s function.

(iii)Theorem (B): The function E(x,y) = ax²+bxy+cy² is positive definite if a>0 and b²-4ac<0 and negative definite if a< 0, b² - 4ac<0. (4)

Notation Adopted

N₁ and N₂ are the populations of the Ammensal and enemy species with natural growth rates a₁ and a₂ respectively.

a₁₂ is rate of increase of the Ammensal due to inhibition by the enemy.

a₂₂ is rate of decrease of the enemy due to insufficient food.

The state variables N₁ and N₂ as well as the model parameters a₁, a₂, and a₂₂ are assumed to be non-negative constants.

3) Basic Equations of the model:

Equation for the growth rate of the Ammensal species (S₁)

$$\frac{dN_1}{dt} = - a_1 N_1 - a_{12} N_1 N_2 \tag{5}$$

Equation for the growth rate of enemy species (S₂)

$$\frac{dN_2}{dt} = -a_2 N_2 - a_{22} N_2^2 \quad (6)$$

Before going to establish the global stability by Liapunov's criteria, we now state the equilibrium states with respective equilibrium points.

4) Equilibrium state and It's stability:

The equilibrium point is obtained as $\bar{N}_1 = 0; \bar{N}_2 = 0$

The corresponding linearised perturbed equations are

$$\frac{dU_1}{dt} = -a_1 U_1 \quad , \quad (7)$$

$$\frac{dU_2}{dt} = -a_2 U_2 \quad (8)$$

The characteristic equation for this system is $(\lambda + a_1)(\lambda + a_2) = 0$
 the roots of which are $-a_1, -a_2$ i.e. both the roots are negative.

Hence the steady state is **stable**.

By solving of (7) and (8), we get $U_1 = U_{10} e^{-a_1 t}; U_2 = U_{20} e^{-a_2 t}$ (9)

CASE 1: $a_1 > a_2$ and $U_{10} > U_{20}$

i.e. the Ammensal species dominates the enemy species in the natural growth rate as well as in its initial population strength.

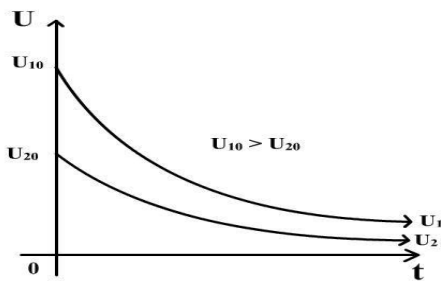


Fig .1

In this case the Ammensal species always out numbers the enemy species. It is evident that both the species are converging asymptotically to the equilibrium point. Hence the state is stable shown in Fig .1. Hence the state is **stable**.

CASE 2: $a_1 > a_2$ and $U_{10} < U_{20}$

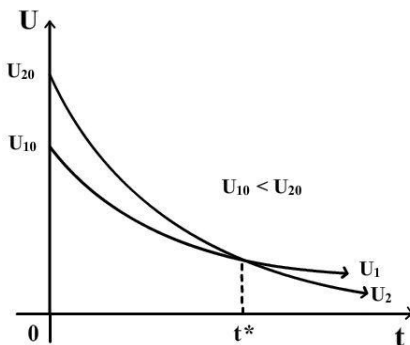


Fig .2

i.e. the Ammensal species dominates the enemy species in the Natural growth rate but its initial strength is less than that of enemy species. Both N_1 and N_2 decline. However the enemy (N_2) dominates over the Ammensal (N_1)

up to the time -instant $t^* = \frac{1}{a_2 - a_1} \log \left(\frac{U_{10}}{U_{20}} \right)$ only

after that the Ammensal species out numbers the enemy species and dominates the enemy species in the natural growth rate as well as in its population strength.

Thus both N_1 and N_2 decline. The enemy (N_2) dominates over the Ammensal (N_1) up to the time –instant t^* (Fig. 2).

CASE 3: $a_1 < a_2$ and $U_{10} < U_{20}$

i.e the enemy species dominates over the Ammensal species in the natural growth rate as well as its initial population strength.

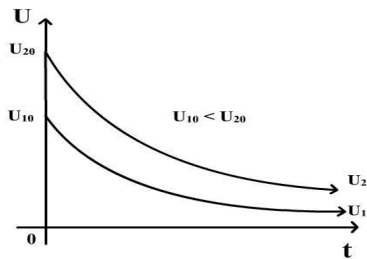


Fig.3

The enemy (N_2) continuous to dominate over the Ammensal (N_1) and both the species converge asymptotically to the equilibrium point. -Fig .3.

CASE 4: $a_1 < a_2$ and $U_{10} > U_{20}$

i.e the enemy species dominates the Ammensal species in the natural growth rate but its initial strength is less than that of Ammensal species .

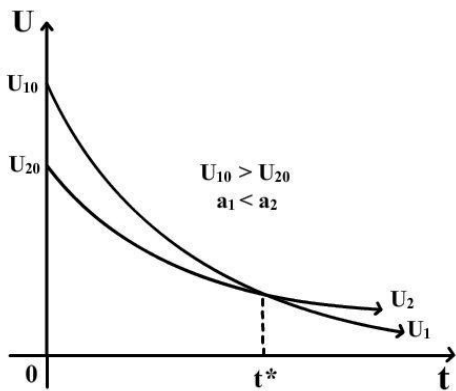


Fig .4

In this case the Ammensal (N_1) out numbers the enemy species till the time

$$t = t^* = \frac{1}{a_2 - a_1} \log \left(\frac{U_{10}}{U_{20}} \right)$$

there after the enemy species (N_2) out numbers the Ammensal (N_1). As $t \rightarrow \infty$ both U_1 and U_2 approach to the equilibrium point as shown in Fig.4. Hence the state is **stable**.

5) Trajectories of Perturbed Species:

The trajectories in $U_1 - U_2$ plane are given by $x^{a_2} = y^{a_1}$ (10)

where $x = \frac{U_1}{U_{10}}$, $y = \frac{U_2}{U_{20}}$ solution curves are illustrated in Fig.5.

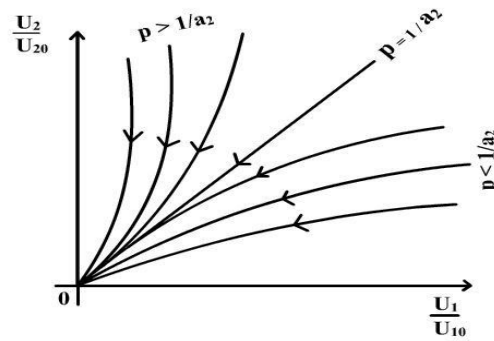


Fig .5

6) The global stability by Liapunov's Analysis:

Linearised basic equations are

$$\frac{dU_1}{dt} = -a_1 U_1 \quad \text{and} \quad \frac{dU_2}{dt} = -a_2 U_2$$

The characteristic equation is

$$(\lambda + a_1) (\lambda + a_2) = 0$$

i.e. $\lambda^2 + (a_1 + a_2) \lambda + a_1 a_2 = 0$

$$\Rightarrow \lambda^2 + p \lambda + q = 0$$

where $P = a_1 + a_2 > 0, q = a_1 a_2 > 0$

(11)

The required conditions for Liapunov's function are satisfied

Let $E(U_1, U_2) = \frac{1}{2} (aU_1^2 + 2b U_1 U_2 + cU_2^2)$

(12)

where $a = \frac{a_2^2 + a_1 a_2}{D}$

$b = 0$

and $c = \frac{a_1^2 + a_1 a_2}{D}$

where $D = pq = (a_1 + a_2)(a_1 a_2) > 0$

(13)

From (8), it is clear that $D > 0$ and $a > 0$

$$D^2(ac - b^2) = b^2 \left[\frac{a_2^2 + a_1 a_2}{D} \cdot \frac{a_1^2 + a_1 a_2}{D} - 0 \right] = (a_2^2 + a_1 a_2)(a_1^2 + a_1 a_2) > 0$$

$$\Rightarrow ac - b^2 > 0 \quad (\text{since } D^2 > 0)$$

i.e., $b^2 - ac < 0$

\therefore The function $E(U_1, U_2)$ is positive definite

(14)

Further $S = \frac{\partial E}{\partial U_1} \cdot \frac{dU_1}{dt} + \frac{\partial E}{\partial U_2} \cdot \frac{dU_2}{dt}$

$$\begin{aligned}
 &= (aU_1+bU_2) (-a_1U_1) + (bU_1 + cU_2)(-a_2U_2) \\
 &= -aa_1U_1^2 - ba_1U_1U_2 - ba_2U_1U_2 - ca_2U_2^2 \\
 &= -aa_1U_1^2 - (ba_1 + ba_2)U_1U_2 - ca_2U_2^2
 \end{aligned} \tag{15}$$

Substituting the values of a, b and c, we obtain

$$\begin{aligned}
 -aa_1U_1^2 &= -\left(\frac{a_2^2 + a_1a_2}{D}\right)a_1U_1^2 = -\left[\frac{(a_1 + a_2)a_1a_2}{D}\right]U_1^2, \\
 -(ba_1 + ba_2)U_1U_2 &= -0(a_1 + a_2)U_1U_2 = 0 \text{ and} \\
 -ca_2U_2^2 &= -\left(\frac{a_1^2 + a_1a_2}{D}\right)a_2U_2^2 \\
 S &= \frac{\partial E}{\partial U_1} \cdot \frac{dU_1}{dt} + \frac{\partial E}{\partial U_2} \cdot \frac{dU_2}{dt} \\
 &= -\left[\frac{(a_1 + a_2)a_1a_2}{D}\right]U_1^2 + 0 - \left[\frac{(a_1 + a_2)a_1a_2}{D}\right]U_2^2 \\
 &= -\frac{D}{D}U_1^2 - \frac{D}{D}U_2^2 = -(U_1^2 + U_2^2)
 \end{aligned} \tag{16}$$

which is negative definite.

So E (U₁, U₂) is a Liapunov function for the linear system.

Next we will prove that E (U₁,U₂) is also a Liapunov function for the non-Linear system

Define F₁(N₁,N₂) = N₁ (-a₁- a₁₂ N₂) and F₂(N₁,N₂) = N₂ (-a₂- a₂₂ N₂)

By putting N₁ = $\bar{N}_1 + U_1$ and N₂ = $\bar{N}_2 + U_2$ in (5) and (6),

$$\begin{aligned}
 \frac{dU_1}{dt} &= (\bar{N}_1 + U_1) (-a_1 - a_{12}(\bar{N}_2 + U_2)) \\
 &= -a_{12}\bar{N}_1U_2 - a_{12}U_1U_2 \\
 \Rightarrow F_1(U_1, U_2) &= \frac{dU_1}{dt} = -a_{12}\bar{N}_1U_2 + f_1(U_1, U_2)
 \end{aligned} \tag{17}$$

where f₁(U₁, U₂) = - a₁₂U₁U₂

$$\begin{aligned}
 \text{similarly } F_2(U_1, U_2) &= \frac{dU_2}{dt} \\
 &= -a_{22}\bar{N}_2U_2 + f_2(U_1, U_2) \text{ where } f_2(U_1, U_2) = -a_{22}U_2^2
 \end{aligned} \tag{18}$$

we have

$$\frac{\partial E}{\partial U_1} F_1 + \frac{\partial E}{\partial U_2} F_2 = -(U_1^2 + U_2^2) + (aU_1 + bU_2) f_1(U_1, U_2) + (bU_1+cU_2) f_2(U_1, U_2)$$

By introducing polar coordinates, we get

$$\frac{\partial E}{\partial U_1} F_1 + \frac{\partial E}{\partial U_2} F_2 = -r^2 + r[(a \cos \theta + b \sin \theta) f_1(U_1, U_2) + (b \cos \theta + c \sin \theta) f_2(U_1, U_2)]$$

Denote largest of the numbers $|a|, |b|, |c|$ by M

Then $|f_1(U_1, U_2)| < \frac{r}{6M}$ and $|f_2(U_1, U_2)| < \frac{r}{6M}$ for all satisfying small $r > 0$

$$\text{so } \frac{\partial E}{\partial U_1} F_1 + \frac{\partial E}{\partial U_2} F_2 < -r^2 + \frac{4Mr^2}{6M} = \frac{-r^2}{3} < 0 \quad (19)$$

Thus $E(U_1, U_2)$ is a positive definite in with the property that

$$\frac{\partial E}{\partial U_1} F_1 + \frac{\partial E}{\partial U_2} F_2 \text{ is negative definite.}$$

\therefore The equilibrium point is asymptotically **stable**.

7 Conclusion: The global stability of this model at fully washed out state is established where the mortality rates of both the species are greater than their birth rates respectively by using Liapunov's stability analysis.

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