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A COVER PROTECTED AMMENSAL WITH CONSTANT HARVESTING RATE AND ENEMY SPECIES PAIR WITH LIMITED RESOURCES

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ABSTACT

The paper explores a mathematical model of a cover protected Ammensal with constant harvesting rate and enemy species pair with limited resources. The Mathematical model is characterized by a couple of first order non-linear ordinary differential equations. All six equilibrium points for this model are identified and their stability criteria are discussed. Solutions for the linearised perturbed equations are found and the results are discussed.

1. INTRODUCTION

Ecological symbiosis can be broadly classified as prey-predation, competition, Mutualism, Commensalism and Ammensalism etc. Of these Ammensalism, a species interaction in sync-ecosystem plays a prominent role in eco-balance in nature. Ammensalism between two species involves one imbedding or restricting the success of the other without getting effected in any manner (positively or negatively) due to the interaction.Research in theoretical ecology was initiated by Lotka [11] and Meyer [12] followed by several mathematicians and ecologists .They contributed their might to the growth of this area of knowledge as reported in the treatises of Kapur [7, 8]. N.C. srinivas [13] studied competitive eco-systems of two and three species with limited and unlimited resources. Later, Lakshminarayan [9], Lakshminarayan and Pattabhi Ramacharyulu [10] studied Prey-predator ecological models with a partial cover for the prey and alternate food for the predator. Recently, Acharyulu [1-6] and Pattabhi Ramacharyulu investigated some results on the stability of an enemy and Ammensal species pair with various resources.

Notation adopted:

N_1, N_2 : The populations of the Ammensal (S_1) and enemy (S_2) species respectively at time t

a_{11} : Rate of decrease of the Ammensal due to insufficient food.

a_{12} : Rate of increase of the Ammensal due to inhibition by the enemy.

a_{22} : Rate of decrease of the enemy due to insufficient food.

h_2 : $a_{22} H_2$ is rate of harvest of the enemy.

K_i : a_i/a_{ii} are the carrying capacities of N_i, i = 1, 2

α : a_{12}/a_{11} is the coefficient of Ammensalism.

b :The constant characterized by the cover which is provided for the Ammensal species (0<b<1)

The state variables N_1 and N_2 as well as the model parameters a_1, a_2, a_{11}, a_{22}, K_1, K_2, α , b , h_1, are assumed to be non-negative constants.

2 Basic Equations

A cover protected Ammensal with constant harvesting rate and enemy species pair with limited resources is characterized by the following pair of coupled non-linear ordinary differential equations.

The equation for the growth Rate of Harvested Ammensal Species (S_1):

$$\frac{dN_1}{dt} = a_{11}(K_1 N_1 - N_1^2 - \alpha (1-b) N_1 N_2 - H_1) \qquad (1)$$

and the equation for the growth Rate of enemy Species (S_2):

$$\frac{dN_2}{dt} = a_{22}(K_2 N_2 - N_2^2) \qquad (2)$$

3 Equilibrium states: The system under investigation has six equilibrium states given by $\frac{dN_1}{dt}=0$ and $\frac{dN_2}{dt} = 0$ and these are classified into four categories.

The equilibrium points are obtained as in the following table.

[redacted]

4 Stability of Equilibrium States:

After linearization, we get $\frac{dU}{dt} = AU$ (3)

$$\text{where } A = \begin{bmatrix} a_{11}(K_1 - 2\overline{N}_1 - \alpha \overline{N}_2) & -a_{11}\,\alpha(1-b)\overline{N}_1 \\ 0 & a_{22}(K_2 - 2\overline{N}_2) \end{bmatrix}$$

The characteristic equation for the system is $\det[A - \lambda I] = 0$ (4)

The equilibrium state is stable only when the roots of the equation (4) are negative, when they are real or have negative real parts in case they are complex

Note: There would be no fully washed out state and also Ammensal washed-out state. However there are three equilibrium states in which the enemy is washed out and three (conditionally) co-existence states.

4.1 Stability of Equilibrium State E_1 :

In this case $A=\begin{bmatrix} -a_{11}\left(K_1-\frac{2H_1}{K_1}\right) & -\alpha(1-b)a_{11}\left(K_1-\frac{H_1}{K_1}\right) \\ 0 & a_{11}K_2 \end{bmatrix}$

The characteristic roots of A are $-a_{11}\left(K_1-\frac{2H_1}{K_1}\right)$, $a_{22}K_2$

and one of these roots is positive, hence the steady state is **unstable**. The equation (3) yields the solution curves.

$$U_1=-M_1 e^{a_{22}K_2 t}+(U_{10}+M_1)\, e^{-a_{11}\left(K_1-\frac{2H_1}{K_1}\right)t},\quad U_2=U_{20}\, e^{a_{22}K_2\, t} \tag{5}$$

where $M_1=\dfrac{\alpha(1-b)\,a_{11}U_{20}\left(K_1-\frac{H_1}{K_1}\right)}{a_{22}K_2+a_{11}\left(K_1-\frac{2H_1}{K_1}\right)}$ and these are illustrated as below.

Case (i): when $U_{10}>U_{20}$ i.e the initial growth rate of Ammensal species is greater than the enemy species

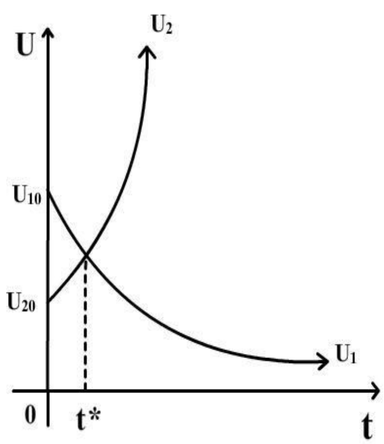

Fig.1

The enemy(S_2) dominates the Ammensal(S_1) in natural growth rate but its initial strength is less than the Ammensal. In this case the enemy outnumbers the Ammensal till the time instant

$$t^*=\frac{1}{a_{22}K_2+a_{11}\left(K_1-\frac{2H_1}{K_1}\right)}\log\left(\frac{U_{10}+M_1}{U_{20}+M_1}\right)$$

after which the enemy is found to be going away from the equilibrium point while the Ammensal species is asymptotic to the equilibrium point. Hence the equilibrium point is **unstable**, as shown in Fig.1.

Case (ii): when $U_{10}<U_{20}$

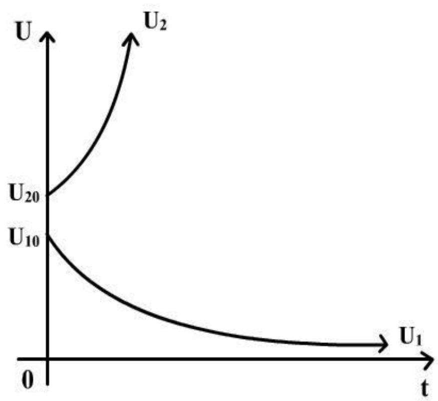

Fig.2

i.e the initial growth rate of Ammensal(S_1) species is less than the enemy(S_2) species.In this case the enemy dominates the Ammensal in natural growth rate as well as in its population strength. The enemy species is noted to be going away from the equilibrium point as shown in Fig.2.

4.1A Trajectories of Perturbed Species:

The trajectories obtained by solving (5) in $U_1 - U_2$ plane can be given by

$$\frac{U_1}{U_{10}}+\frac{M_1}{U_{10}}\left(\frac{U_2}{U_{20}}\right)=\left(1+\frac{M_1}{U_{10}}\right)\left(\frac{U_2}{U_{20}}\right)^{-\beta}, \quad \text{where } \beta=\frac{a_{11}\left(K_1-\frac{2H_1}{K_1}\right)}{a_{22}K_2} \tag{6}$$

and these are illustrated in Fig.3

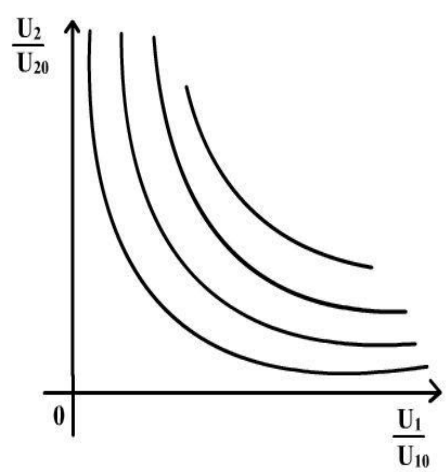

Fig.3

4.2 Stability of Equilibrium State E_2 :

In this case $A=\begin{bmatrix} a_{11}\left(K_1-\frac{2H_1}{K_1}\right) & -\alpha(1-b)a_{11}\left(\frac{H_1}{K_1}\right) \\ 0 & K_2a_{22} \end{bmatrix}$

The characterstic roots of A $a_{11}\left(K_1-\frac{2H_1}{K_1}\right)$, K_2a_{22}

and these are both positive, hence the steady state is **unstable**.

The equation (3) yields the solution curves as

$$U_1=M_2\, e^{k_2 a_{22} t}+U_{10}\, e^{a_{11}\left(k_1-\frac{2H_1}{k_1}\right)t}, \text{ and } U_2=U_{20}\, e^{K_2 a_{22} t} \qquad (7)$$

where $M_2=\dfrac{\alpha(1-b)a_{11}U_{20}\left(\frac{H_1}{K_1}\right)}{a_{11}\left(K_1-\frac{2H_1}{K_1}\right)-a_{22}K_2}$ and these are illustrated below in .Fig.4and Fig.5.

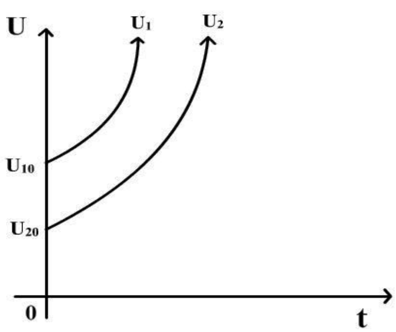

Fig.4

Case (i): when $U_{10}>U_{20}$

i.e the initial growth rate of Ammensal(S_1) species is greater than the enemy(S_2) species.The Ammensal dominates the enemy in natural growth rate as well as in its initial population strength. In this case the Ammensal continuously outnumbers the enemy as shown in Fig.4.

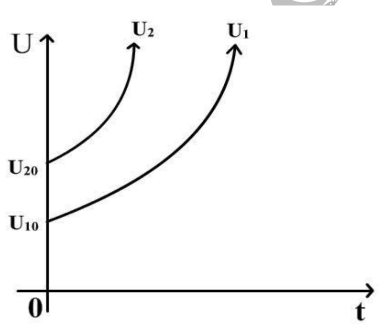

Fig.5

Case (ii): $U_{10}<U_{30}$

i.e the initial growth rate of Ammensal(S_1) species is less than the enemy(S_2) species.The enemy dominates the Ammensal in natural growth rate as well as in its initial population strength. In this case the enemy continuously outnumbers the Ammensal as shown in Fig.5.

4.2A Trajectories of Perturbed Species:

The trajectories obtained by solving (7) in $U_1 - U_2$ plane can be given by

$$\frac{U_1}{U_{10}} = \frac{M_2}{U_{10}}\left(\frac{U_2}{U_{20}}\right) + \left(1 - \frac{M_2}{U_{10}}\right).\left(\frac{U_2}{U_{20}}\right)^q \qquad (8)$$

where $q = \dfrac{a_{11}\left(K_1 - \dfrac{2H_1}{K_1}\right)}{a_{22}K_2}$ which is illustrated in Fig .6

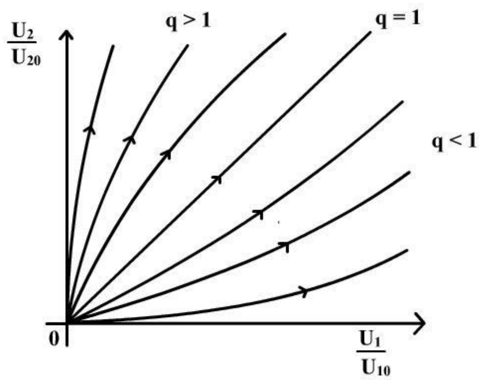

Fig.6

4.3 Stability of Equilibrium State E_3:

In this case A = $\begin{pmatrix} 0 & -\left(\dfrac{\alpha(1-b)\,a_{11}K_1}{2}\right) \\ 0 & K_2 a_{22} \end{pmatrix}$

The characteristic roots of 'A' are 0, K_2a_{22}, hence the steady state is **unstable**.

The equation (3) yields the solution curves.

$$U_1 = -M_3\, e^{K_2a_{22}\,t} + (U_{10} + M_3), \qquad U_2 = U_{20}\, e^{K_2a_{22}\,t} \qquad (9)$$

where $M_3 = \dfrac{\alpha(1-b)\, a_{11}\, U_{20}K_1}{2K_2a_{22}}$ and these are illustrated in Fig.7 and Fig.8

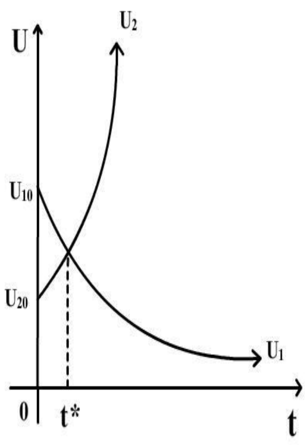

Fig.7

Case (i): when $U_{10}>U_{20}$ i.e the initial growth rate of Ammensal(S_1) species is greater than the enemy(S_2) species.The enemy dominates the Ammensal in natural growth rate but its initial strength is less than the Ammensal. In this case the Ammensal outnumbers the enemy till the time instant $t^*=\dfrac{1}{a_{22}K_2+a_{11}\left(K_1-\dfrac{2H_1}{K_1}\right)}\log\left(\dfrac{U_{10}+M_1}{U_{20}+M_1}\right)$ as shown in Fig.7.

after which the enemy is found to be going away from the equilibrium point while the Ammensal species is asymptotic to the equilibrium point. Hence the equilibrium point is **unstable** as shown in Fig.7.

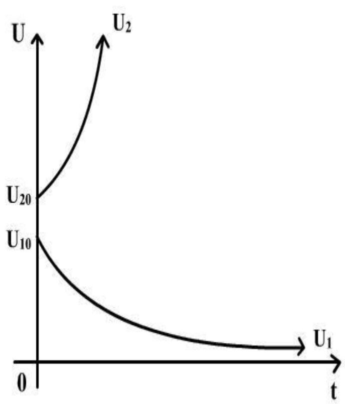

Fig.8

Case (ii): $U_{10}<U_{30}$ i.e the initial growth rate of Ammensal(S_1) species is less than the enemy(S_2) species.In this case the enemy dominates the Ammensal in natural growth rate as well as in its population strength. The enemy species is noted to be going away from the equilibrium point as shown in Fig.8.

4.3A Trajectories of Perturbed Species:

The trajectories obtained by solving (9) in U_1-U_2 plane can be given by

$$\frac{U_1}{U_{10}}+\frac{M_3}{U_{10}}\left(\frac{U_2}{U_{20}}\right)=\left(1+\frac{M_3}{U_{10}}\right) \tag{10}$$

which is illustrated in the following Fig.9

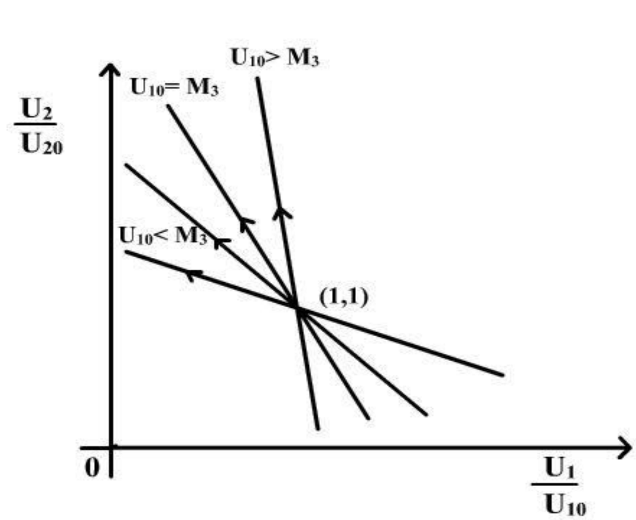

Fig.9

4.4 Equilibrium State E_4:

In this case $A = \begin{bmatrix} -a_{11}\left((K_1-\alpha(1-b)\overline{N}_2) - \frac{2H_1}{K_1-\alpha(1-b)\overline{N}_2}\right) & -\alpha(1-b)\, a_{11}\overline{N}_1 \\ 0 & -a_{22}K_2 \end{bmatrix}$

The characteristic roots of A are $-a_{11}\left((K_1-\alpha(1-b)\overline{N}_2) - \frac{2H_1}{K_1-\alpha(1-b)\overline{N}_2}\right)$, $-a_{22}K_2$ and these are both negative.

Hence the steady state is **stable**.

The equation (3) yields the solution curves:

$$U_1 = M_4\, e^{-a_{22}K_2 t} + (U_{10} - M_4)\, e^{-a_{11}\left((K_1-\alpha(1-b)\overline{N}_2) - \frac{2H_1}{K_1-\alpha(1-b)\overline{N}_2}\right)t},\ U_2 = U_{20}\, e^{-a_{22}K_2 t} \quad (11)$$

where, $M_4 = \dfrac{\alpha(1-b)\, a_{11}\overline{N_1}\, U_{20}}{a_{22}K_2 - a_{11}\left((K_1-\alpha(1-b)\overline{N}_2) - \frac{2H_1}{(K_1-\alpha(1-b)\overline{N}_2)}\right)}$ and these curves are illustrated here under.

when $U_{10} = M_4$,equation (11) becomes

$$U_1 = U_{10}\,.\,e^{-a_{22}K_2 t}\ ;\ U_2 = U_{20}\,.\,e^{-a_{22}K_2 t} \quad (12)$$

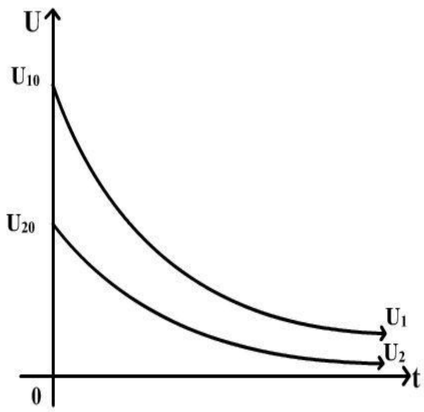

Fig.10

Case (i): $U_{10} > U_{20}$ i.e the initial growth rate of Ammensal(S_1) species is greater than the enemy(S_2) species. Initially the Ammmensal dominates the enemy and it continuous throughout its growth. In this case the Ammensal continuously outnumbers the enemy as shown in Fig.10. However both converge asymptotically to the equilibrium point.

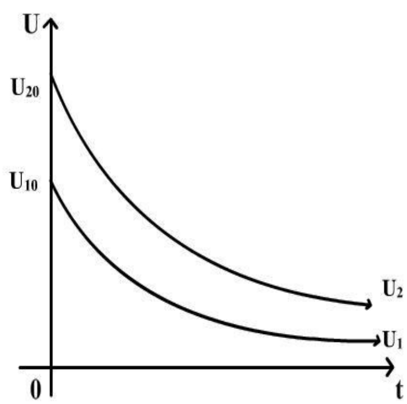

Fig.11

Case (ii): $U_{10} < U_{20}$ i.e the initial growth rate of Ammensal(S_1) species is less than the enemy(S_2) species.Initially the enemy dominates the Ammensal and it continuous to be so throughout its growth rate. In this case the enemy continuously outnumbers the Ammensal as shown in Fig.11.However both converge asymptotically to the equilibrium point.

4.4A Trajectories of Perturbed species:

The trajectories obtained by solving (11) in U_1- U_2 plane can be given by

$$\frac{U_1}{U_{10}} = \frac{M_4}{U_{10}}\left(\frac{U_2}{U_{20}}\right) + \left(1 - \frac{M_4}{U_{10}}\right)\left(\frac{U_2}{U_{20}}\right)^P, \tag{13}$$

where $p = \dfrac{a_{11}\left(\left(K_1 - \alpha(1-b)\overline{N_2}\right) - \dfrac{2H_1}{\left(K_1 - \alpha(1-b)\overline{N_2}\right)}\right)}{a_{22}K_2}$ which are illustrated in Fig.12.

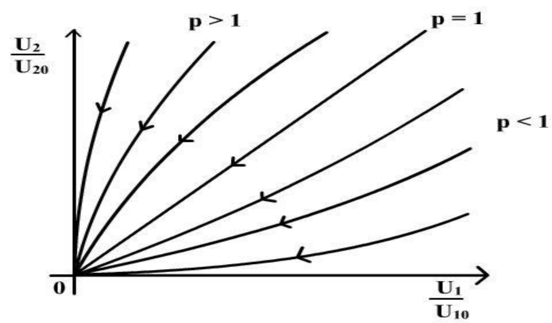

Fig.12

4.5 Stability of Equilibrium state E_5 :

In this case $$A=\begin{bmatrix} a_{11}\left[\left(K_1-\alpha(1-b)\overline{N_2}\right)-\dfrac{2H_1}{K_1-\alpha(1-b)\overline{N_2}}\right] & -\alpha(1-b)\, a_{11}\overline{N_1} \\ 0 & -a_{22}K_2 \end{bmatrix}$$

The characteristic roots of A are

$a_{11}\left(\left(K_1-\alpha(1-b)\overline{N_2}\right)-\dfrac{2H_1}{K_1-\alpha(1-b)\overline{N_2}}\right)$ and $-a_{22}K_2$ since one of these roots is positive, the steady state is **unstable**. The equation (3) yields the solution curves.

$$U_1=M_5\, e^{-a_{22}K_2t}+\left(U_{10}-M_5\right) e^{a_{11}\left(\left(K_1-\alpha(1-b)\overline{N_2}\right)-\frac{2H_1}{K_1-\alpha(1-b)\overline{N_2}}\right)t} \tag{14}$$

$$U_2=U_{20}e^{-a_{22}K_2t} \tag{15}$$

where $M_5=\dfrac{\alpha(1-b)a_{11}\overline{N_1}U_{20}}{a_{22}K_2+a_{11}\left(\left(K_1-\alpha(1-b)\bar{N}_2\right)-\dfrac{2H_1}{K_1-\alpha(1-b)\bar{N}_2}\right)}$ and the solution curves are illustrated.

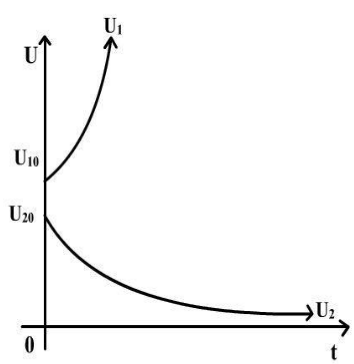

Fig.13

Case (i): $U_{10} > U_{20}$ i.e the initial growth rate of Ammensal(S_1) species is greater than the enemy(S_2) species.In this case the Ammensal dominates the enemy in natural growth rate as well as in its population strength. The Ammensal species is noted to be going away from the equilibrium point as shown in Fig.13 while the enemy species is asymptotic to the equilibrium point.

Case (ii): $U_{10} < U_{20}$ i.e the initial growth rate of Ammensal species is less than the enemy species

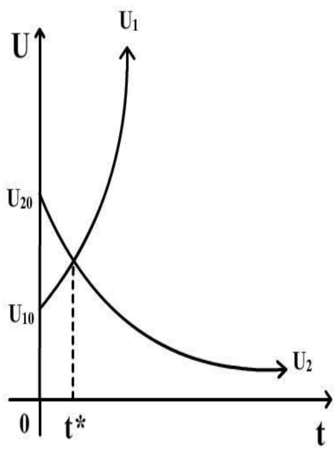

Fig.14

The Ammensal(S_1) dominates the enemy(S_2) in natural growth rate but its initial strength is less than the enemy. In this case the enemy outnumbers the Ammensal till the time- instant

$$t^{*}=\frac{1}{a_{22}K_2+\left[\left(K_1-\alpha(1-b)\overline{N_2}\right)-\frac{2H_1}{\left(K_1-\alpha(1-b)\overline{N_2}\right)}\right]}\log\left[\frac{U_{20}-M_5}{U_{10}-M_5}\right]$$

after which the Ammensal goes away from the equilibrium point while the enemy species is asymptotic to the equilibrium point. Hence the equilibrium point is **unstable,** as shown in Fig.14.

we have noticed that the equilibrium state is conditionally stable when $U_{10} = M_2$

4.5A. Trajectories of Perturbed species:

The trajectories obtained by solving (14) and (15) in $U_1 - U_2$ plane are given by

$$\frac{U_1}{U_{10}}=\frac{M_5}{U_{10}}\left(\frac{U_2}{U_{20}}\right)+\left(1-\frac{M_5}{U_{10}}\right)\left(\frac{U_2}{U_{20}}\right)^{\frac{-a_{11}\left(\left(K_1-\alpha(1-b)\overline{N}_2\right)-\frac{2H_1}{\left(K_1-\alpha(1-b)\overline{N}_2\right)}\right)}{a_{22}K_2}} \qquad (16)$$

which are illustrated in Fig.15

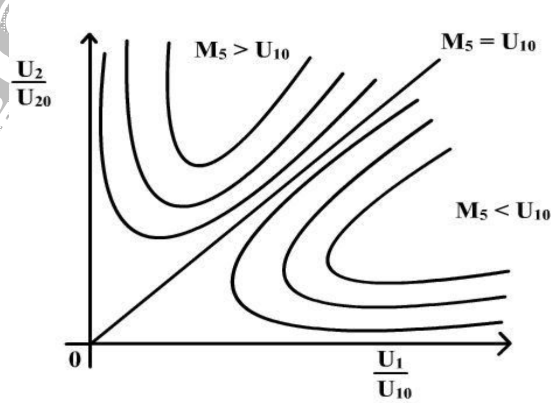

Fig.15

4.6 Equilibrium state E_6 :

In this case $A=\begin{bmatrix} 0 & \frac{-\alpha(1-b)a_{11}}{2}\left(K_1-\alpha(1-b)K_2\right) \\ 0 & -a_{22}\,K_2 \end{bmatrix}$

The characteristic roots of A are 0, $-a_{22}\,K_2$, The equation (3) yields the solution curves.

$$U_1=(U_{10}-M_6)+M_6\,e^{-a_{22}K_2 t}, \qquad U_2=U_{20}\,e^{-a_{22}K_2 t} \tag{17}$$

where $M_6=\dfrac{\alpha(1-b)\,a_{11}U_{20}(K_1-\alpha(1-b)\,K_2)}{2\,a_{22}\,K_2}$

Since one of the characteristic roots is zero, the equilibrium state $\left(\overline{N_1},\overline{N_2}\right)$ is **unstable** in the sense where the perturbations U_1, U_2 do not diminish as $t\to\infty$.However it is noted that U_1, U_2 approach $\left(U_{10}-M_6,0\right)$ i.e., U_1– t curve is asymptotic to the line $U_1=U_{10}-M_6$

and the solution curves are illustrated in Fig.16and Fig.17

Case (i) $\boldsymbol{U_{10} > U_{20}}$i.e the initial growth rate of Ammensal species is greater than the enemy species

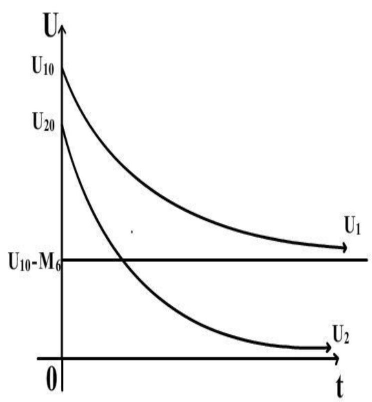

Fig.16

In this case the Ammensal(S_1) species dominates over the enemy(S_2) in it's initial population strength as well as in natural growth rate. Further the Ammensal species is noted to be moving away from equilibrium point asymptotic to the line U_{10}–M_6 while the enemy species is asymptotic to the equilibrium point as shown in Fig16.

Case (ii):when $U_{10} < U_{20}$ i.e the initial growth rate of Ammensal species is less than the enemy species.

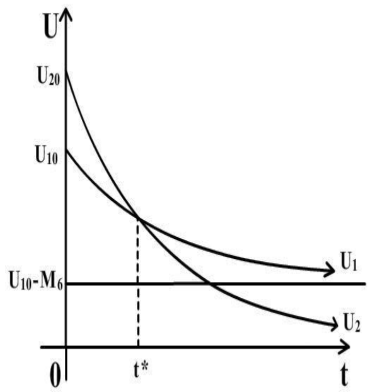

Fig.17

The Ammensal(S_1) species dominates the enemy(S_2) in natural growth rate but its initial population strength is less than the enemy. In this case the enemy outnumbers the Ammensal till the time instant $t^* = \frac{1}{a_{22}K_2}\log\left(\frac{U_{20} - M_6}{U_{10} - M_6}\right)$ after which the Ammensal outnumbers the enemy as shown in Fig.17.

Further the Ammensal species is observed to be moving away from the equilibrium point and is asymptotic to the line .$U_{10} - M_6$ while the enemy species is asymptotic to the equilibrium point. Hence the steady state is **unstable.**

4.6A. Trajectories of Perturbed Species:

The trajectories obtained by solving (17) in $U_1 - U_2$ plane can be given by

$$\frac{U_1}{U_{10}} = \left(1 - \frac{M_6}{U_{10}}\right) + \left(\frac{M_6}{U_{10}}\right)\frac{U_2}{U_{20}}, \tag{18}$$

which are illustrated in Fig.18

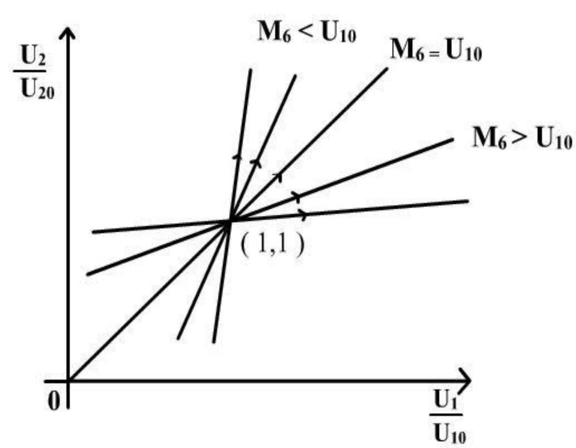

Fig.18

Conclusion: Thus the Stability of a Mathematical model of “A cover protected Ammensal with constant harvesting rate and enemy species pair with limited resources” is established at various equilibrium points and the conclusions are presented in every case there itself.

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