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MODELING AND STABILITY ANALYSIS OF TWO MUTUALLY INTERACTING SPECIES

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Abstract

The present paper concerns with a model of two mutually interacting species with limited resources for first species and unlimited resources for second species. The model is characterized by a coupled system of first order non-linear ordinary differential equations. In all two equilibrium points are identified. If the death rate of the second species is greater than its birth rate, it is found that there are three equilibrium points. The criteria for asymptotic stability have been established for all the equilibrium points. The co-existent state is always stable. The solutions of the linearised basic equations are obtained and their trends are illustrated

Key words: Equilibrium points, Mutualism, Coexistence state, Stability.

1. Introduction:

Mathematical modeling is an important interdisciplinary activity which involves the study of some aspects of diverse disciplines. Biology, Epidemiodology, Physiology, Ecology, Immunology, Bio-economics, Genetics, Pharmocokinetics are some of those disciplines. This mathematical modeling has raised to the zenith in recent years and spread to all branches of life and drew the attention of every one. Mathematical modeling of ecosystems was initiated by Lotka [8] and by Volterra [14]. The general concept of modeling has been presented in the treatises of Meyer [9], Cushing [2], Paul Colinvaux [10], Freedman [3], Kapur [5, 6]. The ecological interactions can be broadly classified as prey-predation, competition, mutualism and so on. N.C. Srinivas [13] studied the competitive eco-systems of two species and three species with regard to limited and unlimited resources. Later, Lakshmi Narayan [7] has investigated the two species prey-predator models. Recently stability analysis of competitive species was investigated by Archana Reddy [1]. Local stability analysis for a two-species ecological mutualism model has been presented by the present authors [11, 12]. Mutualism is any relationship between two species of organisms that benefits both species. Pollination (flowers and insects), seed dispersal (berries and fruit eaten by birds and animals), and lichens (fungus and algae) are examples for mutualism.

Before describing a model, first we make the following assumptions:

N_1 is the population of the first species, N_2, the population of the second species, a_1 is the rate of natural growth of the first species, a_2 is the rate of natural growth of the second species, α_{11} is the rate of decrease of the first species due to insufficient food, α_{12} is the rate of increase of the first species due to interaction with the second species, α_{21} is the rate of increase of the second species due to interaction with the first species. Further note that the variables N_1, N_2 and the model parameters $a_1, a_2, \alpha_{11}, \alpha_{12}, \alpha_{21}$ are non-negative and that the rate of difference between the death and birth rates is identified as the natural growth rate with appropriate sign. The model equations for a two species mutualising are governed by a system of non-linear ordinary differential equations.

2. Basic Equations:

The basic equations are given by

$$\frac{dN_1}{dt} = a_1 N_1 - \alpha_{11} N_1^2 + \alpha_{12} N_1 N_2 \tag{2.1}$$

$$\frac{dN_2}{dt} = -a_2 N_2 + \alpha_{21} N_1 N_2 \tag{2.2}$$

Here we come across three equilibrium states:

I. $\overline{N}_1 = 0; \overline{N}_2 = 0$,

the state in which both the species are washed out .

II. $\overline{N}_1 = \frac{a_1}{\alpha_{11}}; \overline{N}_2 = 0$.

Here the first species (N_1) survives while the second species (N_2) is washed out.

III. $\overline{N}_1 = \frac{a_2}{\alpha_{21}}; \overline{N}_2 = \frac{a_2\alpha_{11} - a_1\alpha_{21}}{\alpha_{12}\alpha_{21}}$.

In this state both the species co-exist and this can exist only when $a_2\alpha_{11} - a_1\alpha_{21} > 0$.

<u>Equilibrium state I</u> (fully washed out state):

To discuss the stability of equilibrium state $\overline{N}_1 = 0; \overline{N}_2 = 0$, we consider small perturbations $u_1(t)$ and $u_2(t)$ from the steady state, i.e. we write

$$N_1 = \overline{N}_1 + u_1(t), \tag{2.3}$$

$$N_2 = \overline{N}_2 + u_2(t) \tag{2.4}$$

Substituting (2.3) and (2.4) in (2.1) and (2.2), we get

$$\frac{du_1}{dt} = a_1 u_1 - \alpha_{11} u_1^2 + \alpha_{12} u_1 u_2$$

$$\frac{du_2}{dt} = -a_2 u_2 + \alpha_{21} u_1 u_2$$

After linearization, we get

$$\frac{du_1}{dt} = a_1 u_1 \tag{2.5}$$

and

$$\frac{du_2}{dt} = -a_2 u_2 \tag{2.6}$$

The characteristic equation is

$$(\lambda - a_1)(\lambda + a_2) = 0$$

One root of this equation is $\lambda_1 = a_1$ which is positive and the other root is $\lambda_2 = -a_2$ which is negative. Hence the equilibrium state is **unstable.**

The solutions of equations (2.5) and (2.6) are

$$u_1 = u_{10}\, e^{a_1 t} \tag{2.7}$$

$$u_2 = u_{20}\, e^{-a_2 t} \tag{2.8}$$

where u_{10}, u_{20} are the initial values of u_1 and u_2. The solution curves are illustrated in figures 1 and 2.

Case 1: $u_{10} > u_{20}$ i.e. initially the first species dominates the second species.

We notice that the first species is going away from the equilibrium point while the second species approaches asymptotically to the equilibrium point. Hence the state is unstable.

Case 2: $u_{10} < u_{20}$ i.e. initially the second species dominates the first species.

In this case the second species out numbers the first species till the time,

$$t = t^* = \frac{ln\,\{u_{20} / u_{10}\}}{(a_1 + a_2)}$$

after that the first species out numbers the second species and grows indefinitely while the second species approaches asymptotically to the equilibrium point. Hence the state is unstable. Further the trajectories in the (u_1, u_2) plane are given by

$$\left[\frac{u_1}{u_{10}}\right]^{a_2} = \left[\frac{u_2}{u_{20}}\right]^{-a_1}$$

<u>Equilibrium state II</u> (N_1 exists while N_2 is washed out):

We have

$$\overline{N}_1 = \frac{a_1}{\alpha_{11}}; \overline{N}_2 = 0$$

Substituting (2.3) and (2.4) in (2.1) and (2.2), we get

$$\frac{du_1}{dt} = -a_1 u_1 - \alpha_{11} u_1^2 + \alpha_{12} u_1 u_2 + \frac{a_1 \alpha_{12} u_2}{\alpha_{11}}$$

$$\frac{du_2}{dt} = -a_2 u_2 + \alpha_{21} u_1 u_2 + \frac{a_1 \alpha_{21} u_2}{\alpha_{11}}$$

After linearization, we get

$$\frac{du_1}{dt} = -a_1 u_1 + \frac{a_1 \alpha_{12} u_2}{\alpha_{11}} \tag{2.9}$$

$$\frac{du_2}{dt} = \left[\frac{a_1 \alpha_{21}}{\alpha_{11}} - a_2\right] u_2 \tag{2.10}$$

The characteristic equation is

$$(\lambda+a_1)\ \{\lambda\text{-}[\frac{a_1\alpha_{21}}{\alpha_{11}}-a_2]\}=0 \qquad (2.11)$$

One root of this equation (2.11) is $\lambda_1 = -a_1$ which is negative.

Case A: When $\frac{a_1}{a_2} > \frac{\alpha_{11}}{\alpha_{21}}$,

the other root of equation (2.11) is $\lambda_2 = \frac{a_1\alpha_{21}}{\alpha_{11}} - a_2$ which is positive. Hence the equilibrium state is **unstable.**

The trajectories are given by

$$u_1 = \frac{1}{\gamma_1}\left[u_{20}a_1\alpha_{12}e^{\lambda_2 t} + \{u_{10}\gamma_1 - u_{20}a_1\alpha_{12}\}e^{-a_1 t}\right] \qquad (2.12)$$

$$u_2 = u_{20}e^{\lambda_2 t} \qquad (2.13)$$

where

$$\gamma_1 = a_1[\alpha_{11}+\alpha_{21}] - a_2\alpha_{11}$$

The solution curves are illustrated in **figures 3&4.**

CASE 1: For $u_{10} < u_{20}$, we have

In this case the second species is noted to be going away from the equilibrium point while the first species would become extinct at the instant

$$t_1^* = \frac{1}{(\lambda_2+a_1)} ln\left[\frac{u_{20}\alpha_{12}a_1 - u_{10}\gamma_1}{u_{20}\alpha_{12}a_1}\right]$$

As such the state is **unstable**.

CASE 2: If $u_{10} > u_{20}$, we have

Here the first species out numbers the second species till the time,

$$t = t^* = \frac{1}{\lambda_2+a_1} ln\left\{\frac{u_{10}\alpha_{11}(\lambda_2+a_1) - u_{20}a_1\alpha_{12}}{u_{20}[\alpha_{11}(\lambda_2+a_1) - a_1\alpha_{12}]}\right\}$$

there after the second species out numbers the first species. And also the second species is noted to be going away from the equilibrium point while the first species would become extinct at the instant

$$t_1^* = \frac{1}{(\lambda_2+a_1)} ln\left[\frac{u_{20}\alpha_{12}a_1 - u_{10}\gamma_1}{u_{20}\alpha_{12}a_1}\right]$$

As such the state is **unstable**.

Case B: When $\frac{a_1}{a_2} < \frac{\alpha_{11}}{\alpha_{21}}$

One root of the equation (2.11) is $\lambda_1 = -a_1$ which is negative and the other root is

$\lambda_2 = \frac{a_1\alpha_{21}}{\alpha_{11}} - a_2$ which is negative.

As the roots of the equation (2.11) are both negative, the equilibrium state is **stable.**

The trajectories in this case are the same as in (2.12) and (2.13).
That is

$$u_1 = \frac{1}{\gamma_1}\left[u_{20} a_1 \alpha_{12} e^{\lambda_2 t} + \{u_{10}\gamma_1 - u_{20} a_1 \alpha_{12}\} e^{-a_1 t}\right]$$

$$u_2 = u_{20} e^{\lambda_2 t}$$

where

$$\gamma_1 = a_1[\alpha_{11} + \alpha_{21}] - a_2\alpha_{11}$$

The solution curves are illustrated in **figures 5&6.**

CASE 1: $u_{10} < u_{20}$ i.e. initially the second species dominates the first species.
In this case the second species always out numbers the first species. It is evident that both the species converging asymptotically to the equilibrium point. Hence the state is **stable**.
CASE 2: $u_{10} > u_{20}$ i.e. initially the first species dominates the second species.

Here the first species out numbers the second species till the time,

$$t = t^* = \frac{1}{\lambda_2 + a_1} ln \left\{\frac{u_{10}\alpha_{11}(\lambda_2 + a_1) - u_{20} a_1 \alpha_{12}}{u_{20}[\alpha_{11}(\lambda_2 + a_1) - a_1\alpha_{12}]}\right\}$$

there after the second species out numbers the first species. As $t \to \infty$ both u_1 & u_2 approach to the equilibrium point. Hence the state is **stable.** Further the trajectories in the (u_1, u_2) plane are given by

$$(q_1 - 1)u_1 = c\, u_2^{\,q_1} + p_1 u_2$$

where

$$p_1 = \frac{a_1\alpha_{12}}{a_2\alpha_{11} - a_1\alpha_{21}};$$

$$q_1 = \frac{a_1\alpha_{11}}{a_2\alpha_{11} - a_1\alpha_{21}}$$

and c is an arbitrary constant.
The solution curves are illustrated in figure 7.

<u>Equilibrium state III</u> (coexistence state):

We have

$$\overline{N}_1 = \frac{a_2}{\alpha_{21}};\ \overline{N}_2 = \frac{a_2\alpha_{11} - a_1\alpha_{21}}{\alpha_{12}\alpha_{21}} \text{ wherein } a_2\alpha_{11} - a_1\alpha_{21} > 0$$

Substituting (2.3) and (2.4) in (2.1) and (2.2), we get

$$\frac{du_1}{dt} = -\alpha_{11}u_1^{\,2} + \alpha_{12}u_1u_2 - \alpha_{11}\overline{N}_1u_1 + \alpha_{12}\overline{N}_1u_2$$

$$\frac{du_2}{dt} = \alpha_{21}u_1\overline{N}_2 + \alpha_{21}u_1u_2$$

After linearization, we get

$$\frac{du_1}{dt} = -\alpha_{11}\overline{N}_1u_1 + \alpha_{12}\overline{N}_1u_2 \qquad (2.14)$$

and

$$\frac{du_2}{dt} = \alpha_{21} u_1 \overline{N}_2 \tag{2.15}$$

The characteristic equation is

$$\lambda^2 + \alpha_{11}\overline{N}_1\lambda - \alpha_{12}\alpha_{21}\ \overline{N}_1\overline{N}_2 = 0$$

One root of this equation is positive and the other root is negative. Hence the equilibrium state is **unstable**.

The trajectories are given by

$$u_1 = \left[\frac{u_{10}\lambda_1 + u_{20}\alpha_{12}\overline{N}_1}{\lambda_1 - \lambda_2}\right] e^{\lambda_1 t} + \left[\frac{u_{10}\lambda_2 + u_{20}\alpha_{12}\overline{N}_1}{\lambda_2 - \lambda_1}\right] e^{\lambda_2 t}$$

$$u_2 = \left[\frac{u_{20}(\lambda_1 + \alpha_{11}\overline{N}_1) + u_{10}\alpha_{21}\overline{N}_2}{\lambda_1 - \lambda_2}\right] e^{\lambda_1 t} + \left[\frac{u_{20}(\lambda_2 + \alpha_{11}\overline{N}_1) + u_{10}\alpha_{21}\overline{N}_2}{\lambda_2 - \lambda_1}\right] e^{\lambda_2 t}$$

The curves are illustrated in **figures 8&9.**

Case 1: $u_{10} > u_{20}$ i.e. initially the first species dominates the second species.

In this case, the first species is noted to be going away from the equilibrium point while the second species approaches asymptotically to the equilibrium point. Hence the state is unstable.

Case 2: $u_{10} < u_{20}$ i.e. initially the second species dominates the first species.

In this case the second species out numbers the first species till the time,

$$t = t^* = \frac{1}{\lambda_2 - \lambda_1}\ ln\left[\frac{(b_2 - \lambda_1)u_{10} + (a_3 - b_1)u_{20}}{(b_2 - \lambda_2)u_{10} + (a_4 - b_1)u_{20}}\right]$$

where

$$b_1 = \alpha_{12}\overline{N}_1; \quad b_2 = \alpha_{21}\overline{N}_2;$$

$$a_3 = \lambda_1 + \alpha_{11}\overline{N}_1; \quad a_4 = \lambda_2 + \alpha_{11}\overline{N}_1$$

after that the first species out numbers the second species and grows indefinitely while the second species approaches asymptotically to the equilibrium point. Hence the state is unstable. Further the trajectories in the (u_1, u_2) plane are given by

$$[u_2^{(a-1)(v_1-v_2)}]d = \frac{(u_1 - u_2 v_1)^{av_1}}{(u_1 - v_2 u_2)^{av_2}}$$

where v_1 and v_2 are roots of the quadratic equation av^2+bv+c=0 with a=$\alpha_{21}\overline{N}_2$; b=$\alpha_{11}\overline{N}_1$; $c = -\alpha_{12}\overline{N}_1$ and d is an arbitrary constant.

Trajectories:

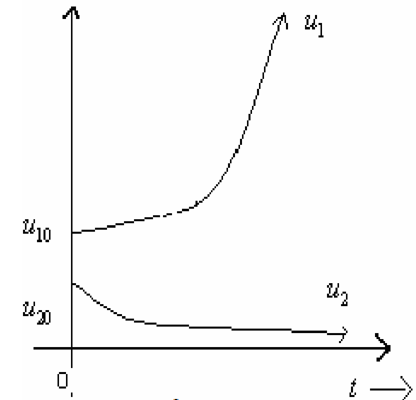

Figure 1

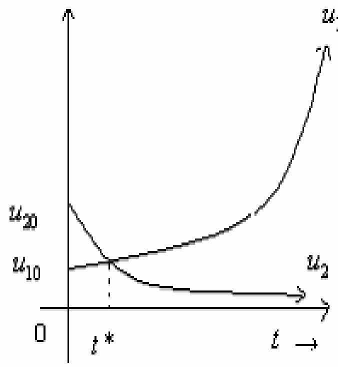

Figure 2

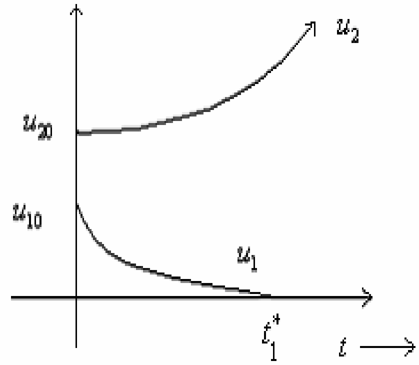

Figure 3

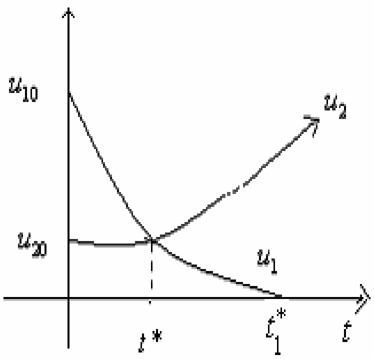

Figure 4

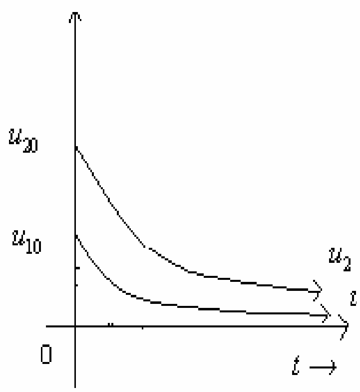

Figure 5

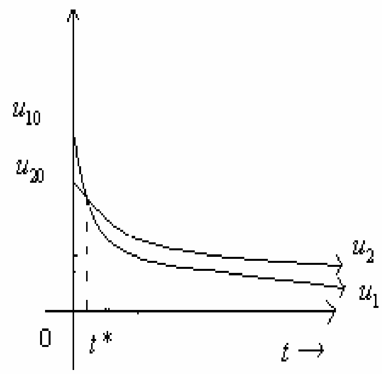

Figure 6

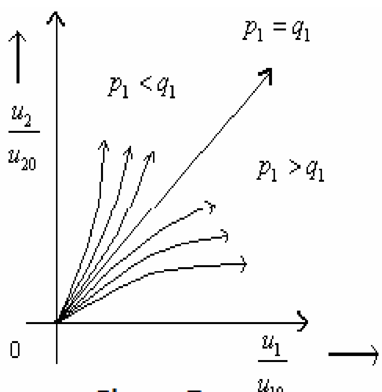

Figure 7

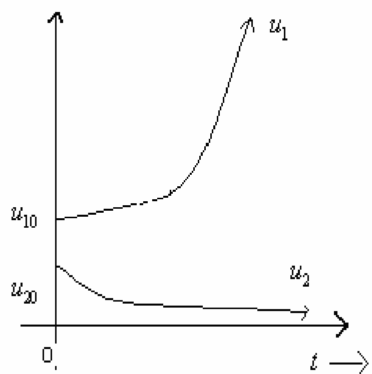

Figure 8

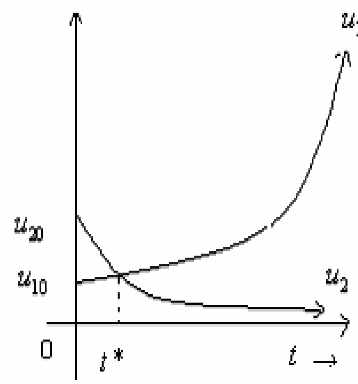

Figure 9

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