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INITIAL VALUE METHOD FOR SOLVING SECOND ORDER SINGULARLY PERTURBED TWO POINT BOUNDARY VALUE PROBLEMS

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Abstract: Initial value method is presented for solving singularly perturbed two point boundary value problems . The presence of first derivative term leads to a boundary layer region nearer the left end point of the interval . In this method, the approximate solution is obtained by solving the reduced problem and an initial value problem associated with the given singularly perturbed problem numerically. The reduced problem is solved by Runge Kutta method of order four and the other initial value problem is solved by a stiff method(exponentially fitted method of Doolan et al.,) of order one. The method do not require the matrix inversion for the numerical convergence. The method presented in this paper is a modified form of the method of Gasparo and Macconi. The error estimates for the numerical convergence of the method of Gasparo and Macconi and the method presented in this paper are derived. Numerical results are given in this paper to demonstrate the applicability of the initial value method.

Keywords: singular perturbation problems, exponentially fitted, uniformly convergent, asymptotic expansion, finite difference schemes, initial value method.

1. Introduction

The numerical treatment for singular perturbation problems have always been far from trivial, because of the boundary layer behavior of the solution. These problems occur frequently in fluid mechanics, elasticity and other branches of applied mathematics, science and engineering. A few notable problems are boundary layer problems, WKB problems, convective heat transport problems with large Peclet number, etc. The area of singular perturbations is a field of increasing interest to applied mathematicians. To be specific, we consider the following singular perturbation problem(SPP):

$$Lu(x) \equiv -\varepsilon u''(x) + a(x)\, u'(x) + b(x)\, u(x) = f(x), \qquad (1a)$$

for $c < x < d$ with

$$u(c) = \varphi_1 \text{ and } u(d) = \varphi_2 \qquad (1b)$$

where ε is a small parameter ($c < \varepsilon << d$), φ_1, φ_2 are given constants, a(x), b(x) and f(x) are assumed to be sufficiently continuously differentiable functions in

[c, d], and $|a(x)| \geq a(0) > 0$, $b(x) \geq \beta > 0$ on [c, d], where β is some positive constant. Under these assumptions SPP(1a,b) has a unique solution u(x) which, in general, displays a boundary layer of width O(ε) at x=c for small values of ε [1-19].

Uniformly convergent finite difference schemes for the SPP(1a,b) have been examined by various authors [3,6,9,12,14,18]. All these schemes use constant mesh size and it is impractical if one wants to find local behavior of the solution in the neighborhood of ε , where ε is small.

Pearson[15] was perhaps the first to attempt something like net adjustments in finite difference schemes for the boundary value problem with first derivative term . Roberts[16] proposed a boundary value technique and introduced the idea of inner and outer region problems for the domain [0, 1]. Such type of technique is also discussed in [7-9,19]. Other works include Bender[1], Neyfeh]11] and O'Malley[13].

Initial value method is presented in [4,5,10]. This method do not involve matrix inversion. It is observed that from Table 1, for h=0.1 and $\varepsilon = 1$, the absolute error is of $O(10^{-1})$. From Tables 2, for small values of ε the numerical solution behaves as if the exact solution behaves. The error estimate is also not derived in [5] for the initial value method.

An attempt have been made in this paper to device an initial valve method to rectify the behavior of the numerical solution obtained by the method of Gasparo and Macconi[5].

In section 2 the initial value method of Gasparo and Macconi[5] is given. The error estimate of the numerical solutions of the method is derived in the same section . In section 3 the description of the new initial value method is presented. The error estimate of the numerical solutions of the method is derived in the same section . The numerical experimental results are presented in section 4.

2. Initial Value method of Gasparo and Macconi

Consider the SPP(1a,b). as the original problem. We split the original problem into two initial value problems, namely, the initial value problem connecting the left boundary layer region and the outer region problem.

The initial value problems are

$$\varepsilon v'(x) + [a(x) - \varepsilon b(x)/a(x)]v(x) = 0, \quad x \in [c,d], \tag{2a}$$

$$v(c) = 1 \tag{2b}$$

and

$$a(x)w'(x) + b(x)w(x) = f(x), \quad x \in [c,d], \tag{3a}$$

$$w(d) = \varphi_2 . \tag{3b}$$

It is shown that in [5]

$$U(x) = w(x) + [\varphi_1 - w(c)]v(x) + O(\varepsilon). \tag{4}$$

The numerical method of Gasparo and Macconi[5] is:

$$u_i = w_i + [\varphi_1 - w(c)]v_i . \quad 0 < i < N-1, \tag{5a}$$

and

$$u_0 = \varphi_1, u_N = \varphi_2 . \tag{5b}$$

2.1. Error estimate of u(x) - U (x).

Using maximum principle we derive error estimates for u(x) - U (x) , where u(x) is the exact solution and U(x) is the approximate solution of (1a.b) as in (4) obtained by Gasparo and Macconi[5] in Theorem 2. The maximum principle is stated as follows in Theorem 1 [3, 18]:

Theorem 1.

Let v *be any smooth function and* L *be the operator defined as in* (1a).

(i) *if* $v(c) \geq 0$, $v(d) \geq 0$, *and* $Lv(x) \leq 0$, *for* $x \in (c, d)$, *then we have* $v(x) \geq 0$, *for all* $x \in [c, d]$,

(ii) *for all* $x \in [c, d]$, *we have*

$| v(x) | \leq C \max(| v(c) |, | v(d) |, \max | Lv(y) |)$, $y \in [c, d]$ *and* $C > 0$.

Proof : See Doolan et al., [3] for proof of Theorem 1.

Theorem 2.

Let u *and* U *be the solutions of the SPP*(1a,b) *and* (4) *respectively. Then , for all* $x \in [c, d]$,

$$| u(x) - U(x) | \leq C \varepsilon \qquad (6)$$

where C *is independent of* ε .

Proof. For all c x < d, we have

$L [u(x) - U(x)] = L u(x) - L U(x) = f(x) - [f(x) + O(\varepsilon)] = O(\varepsilon)$.

For x=c, $u(c) - U(c) = \varphi_1 - \varphi_1 = 0$.

And for x = d, $u(d) - U(d) = \varphi_2 - \varphi_2 = 0$.

Using maximum principle, for all $x \in [c, d]$ we have

$| u(x) - U(x) | \leq | L u(x) - L U(x) | \leq C \varepsilon$.

2.2.. Error estimates-Numerical solution.

Using the discrete maximum principle we derive error estimates for the numerical solutions of Gasparo and Macconi[5] in Theorem 4. The discrete maximum principle is stated as follows in Theorem 3 [3, 18]:

Theorem 3.

Let v_i *be a mesh function and* L^h *be the operator associated with the scheme* (5a,b).

(i) *if* $v_0 \geq 0$, $v_N \geq 0$, *and* $L^h v_i \leq 0$, *for all* $1 \leq i \leq N-1$, *then we have* $v_i \geq 0$, *for all* $0 \leq i \leq N$,

(ii) *for all* $0 \leq i \leq N$, *we have*

$| v(x) | \leq C \max(| v_0 |, | v_N |, \max | L^h v_j |)$, for all $0 \leq j \leq N$ *and* $C > 0$.

Proof: See Doolan et al., [3] for proof of Theorem 3.

Theorem 4.,

Let u(x) *be the solution of* (1a,b) *and* u_i *be the numerical solution of the* (1a,b), *as defied in* (5a,b). *Then we have for all* $x \in [c,d]$ *and* $0 \le i \le N$,

$$|u(x_i) - u_i| \le C[\varepsilon + h^4 + h^n],\ n \ge 1 \qquad (7)$$

where C *is independent of* i, h *and* ε.

Proof: For i=0, $u(x_0) - u_0 = \varphi_1 - \varphi_1 = 0$.

For i=N, $u(x_N) - u_N = \varphi_2 - \varphi_2 = 0$.

And , for $1 < i < N-1$,

the difference between the exact and approximate solution is , for $1 < i < N-1$,

$$\begin{aligned} u(x_i) - u_i &= u(x_i) - U(x_i) + U(x_i) - u_i \\ &= [u(x_i) - U(x_i)] + [w(x_i) + [\varphi_1 - w(c)]v(x_i) - u_i] \\ &= [u(x_i) - U(x_i)] + [w(x_i) + [\varphi_1 - w(c)]v(x_i) - w_i + \varphi_1 - w(c)]v_i] \\ &= [u(x_i) - U(x_i)] + [w(x_i) - w_i] + [\varphi_1 - w(c)][v(x_i) - v_i]. \end{aligned} \qquad (8)$$

From (6), it is clear that

$$|u(x_i) - U(x_i)| \le C\varepsilon. \qquad (9)$$

On applying Runge kutta method of order four, we have

$$|w(x_i) - w_i| \le Ch^4. \qquad (10)$$

On applying a stiff method of order n, $n \ge 1$, we have

$$|v(x_i) - v_i| \le Ch^n, n \ge 1. \qquad (11)$$

Finally,1

$$\begin{aligned} |u(x_i) - u_i| &\le |u(x_i) - U(x_i)| + |w(x_i) - w_i| + |\varphi_1 - w(c)||v(x_i) - v_i| \\ &\le C\varepsilon + Ch^4 + Ch^n, n > 1 \\ &\le C[\varepsilon + h^4 + h^n],\ n \ge 1. \end{aligned} \qquad (12)$$

Using stability result[3, 18], we have, for all $x \in [c,d]$ and $0 \le i \le N$

$$|u(x_i) - u_i| \le \le C[\varepsilon + h^4 + h^n],\ n \ge 1.$$

3. Initial Value method

Consider the SPP(1a,b) as the original problem. We split the original problem into two initial value problems, namely, the initial value problem connecting the left boundary layer region and the outer region problem.

The initial value problems are

$$\varepsilon v'(x) + a(x)v(x) = 0,\ x \in [c,d], \qquad (13a)$$

$$v(c) = 1 \qquad (13b)$$

and

$$a(x)w'(x) + b(x)w(x) = f(x),\ x \in [c,d], \qquad (14a)$$

$$w(d) = \varphi_2. \qquad (14b)$$

It is shown that in [3,18]

$$U(x) = w(x) + [\varphi_1 - w(c)]v(x) + O(\varepsilon). \qquad (15)$$

The numerical solution is defined as

$$u_i = w_i + [\varphi_1 - w(c)]v_i . \quad 0 < i < N-1, \tag{16a}$$

and

$$u_0 = \varphi_1 , u_N = \varphi_2 . \tag{16b}$$

3..1.Error estimate of u(x) - U (x).

Using maximum principle we derive error estimates for u(x) - U (x) , where u(x) is the exact solution and U(x) is the approximate solution of (1a.b) as in (15) in Theorem 5.

Theorem 5.

Let u *and* U *be the solutions of the SPP*(1a,b) *and* (15) *respectively. Then , for all* $x \in [c, d]$,

$$| u(x) - U(x) | \le C \varepsilon \tag{17}$$

where C *is independent of* ε .

Proof. For all c x < d, we have

$L [u(x) - U(x)] = L u(x) - L U(x) = f(x) - [f(x) + O(\varepsilon)] = O(\varepsilon)$.

For x=c, $u(c) - U(c) = \varphi_1 - \varphi_1 = 0$.

And for x = d, $u(d) - U(d) = \varphi_2 - \varphi_2 = 0$

Using maximum principle, for all $x \in [c,d]$, we have

$| u(x) - U(x) | \le | L u(x) - L U(x) | \le C \varepsilon$.

3.2.. Error estimates-Numerical solutions

Using the discrete maximum principle we derive error estimates for the numerical solutions in Theorem 6.

Theorem 6.,

Let u(x) *be the solution of* (1a,b) *and* u_i *be the numerical solution of the* (1a,b),*as defied* in (16a,b). *Then we have for all* $x_i \in$ [c,d] *and* $0 \le i \le N$,

$$| u(x_i) - u_i | \le C[\varepsilon + h^4 + \min(h^n, \varepsilon)], \quad n \ge 1 \tag{18}$$

where C *is independent of* I, h *and* ε.

Proof: For i=0, $u(x_0) - u_0 = \varphi_1 - \varphi_1 = 0$.

For i=N, $u(x_N) - u_N = \varphi_2 - \varphi_2 = 0$.

And , for 1 < i < N-1,

the difference between the exact and approximate solution is , for 1 < i < N-1,

$$\begin{aligned} u(x_i) - u_i &= u(x_i) - U(x_i) + U(x_i) - u_i \\ &= [u(x_i) - U(x_i)] + [w(x_i) + [\varphi_1 - w(c)]v(x_i) - u_i] \\ &= [u(x_i) - U(x_i)] + [w(x_i) + [\varphi_1 - w(c)]v(x_i) - w_i + \varphi_1 - w(c)]v_i] \\ &= [u(x_i) - U(x_i)] + [w(x_i) - w_i] + [\varphi_1 - w(c)][v(x_i) - v_i]. \end{aligned} \tag{19}$$

From (17), it is clear that

$$| u(x_i) - U(x_i) | \le C\varepsilon. \tag{20}$$

On applying Runge kutta method of order four, we have

$$| w(x_i) - w_i | \le Ch^4. \quad (21)$$

On applying a stiff method (exponentially fitted method of Doolan et al.,[3])of order n, n≥1, we have

$$| v(x_i) - v_i | \le C \min(h^n, \varepsilon), n \ge 1. \quad (22)$$

Finally,1

$$|u(x_i) - u_i| \le |u(x_i) - U(x_i)| + | w(x_i) - w_i | + | \varphi_1 - w(c)| | v(x_i) - v_i |$$
$$\le C\varepsilon + Ch^4 + C \min(h^n, \varepsilon) \quad , n \ge 1$$
$$\le C[\varepsilon + h^4 + \min(h^n, \varepsilon)], n \ge 1. \quad (23)$$

Using stability result[3, 18], we have, for all $x \in [c,d]$ and $0 \le i \le N$

$$|u(x_i) - u_i| \le \le C[\varepsilon + h^4 + \min(h^n, \varepsilon)], n \ge 1.$$

Remark.. On applying the initial value method to the example 1, it is observed from Table 3, the absolute error is of $O(10^{-2})$ for $\varepsilon = 1$ and $h = 0.1$. And from Tables 3 and 4, for small values of ε the numerical solution converges to the order $O(\varepsilon + h^4 + \min(h, \varepsilon))$. The initial value problem (13a,b) is solved by a stiff method (exponentially fitted method of Doolan et al.,)of order one and the initial value problem(14a,b) is solved by Runge Kutta method of order four.

4. Numerical experiment

To demonstrate the applicability of the initial value l method , we have implemented it on a SPP. Computed results are tabulated in Tables 1-4. In all the tables h=0.1 a large mesh size is considered. The initial value methods discussed in section 2 and 3 are applied to the following test example

Example 1.

Consider the following non-homogeneous SPP from [5]

$$\varepsilon u''(x) + (1+x)^2 u'(x) - 2(1+x)u(x) = 0.5 [(1+x)(3-x) + \varepsilon/2] \exp(-x/2), \quad 0 < x < 1,$$

$$u(0) = 0 , \; u(1) = \exp(-1/2) - \exp(-[7/3]\varepsilon),$$

whose exact solution is

$$u(x) = \exp(-x/2) - \exp(-[x^3 + 3x^2 + 3x] / (3\varepsilon)).$$

The numerical results are presented in Tables 1,2,3 and 4 for different values of ε. In the tables nodal points(x_i), approximate solution(u_i), exact solution($u(x_i)$) and the absolute error ($u(x_i) - u_i$) are tabulated.

In the method of Gasparo and Macconi[5], the numerical solution v_i of (2a,b) is obtained by a stiff method(exponentially fitted method of Doolan et al.,[3])of order one and w_i of (3a,b) by Runge Kutta method of order four for simplicity. It is

observed from Table 1, the absolute error is of $O(10^{-1})$ for h=0.1 and $\varepsilon = 1$. But from Tables 1 and 2, for small values of ε the numerical solution converges to the order $O(\varepsilon + h^4 + \min(h,\varepsilon))$.

In the method discussed in section 3, the numerical solution v_i of (13a,b) is obtained by a stiff method of order one and w_i of (14a,b) by Runge-Kutta method of order four for simplicity. It is observed from Table 3, the numerical solution behaves for h=0.1 and $\varepsilon = 1$ as the exact solution behaves and the absolute error is of $O(10^{-2})$. And from Tables 3 and 4, for small values of ε the numerical solution converges to the order $O(\varepsilon + h^4 + \min(h, \varepsilon))$.

One can apply higher order methods to solve the stiff initial value problem in the present initial value method to get better accuracy.

5. Discussion and conclusion

We have presented a practical method, exactly implemented on a computer to solve singularly perturbed two point boundary value problems with a fist derivative term. No iteration is involved for the numerical convergence in the method. We have demonstrated that the initial value method approximates the exact solution well, with a standard example. The method presented in this paper is superior than the method of Gasparo and Macconi[5] when ε takes the value one and moderate values. And for small values of ε, the method presented in this paper is also superior.

To get more mesh points inside the boundary layer with an a priori chosen accuracy, even though this method is not uniformly convergent, one can use the method in[19].

It can be observed from Tables that the present method approximates the exact solution of the singularly perturbed problem very well.

All computations were performed in Pascal single precision on a Micro Vax II computer at Bharathidasan University, Tiruchirapalli-620 024, Tamil Nadu, India.

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Table 1.

h= 1.0000000000E-01

ε = 1.0000000000E+00

x_i	u_i	$u(x_i)$	$u(x_i) - u_i$
0.00000E+00	0.00000E+00	0.00000E+00	0.00000E+00
1.00000E-01	5.47424E-02	5.56937E-02	9.51332E-04
2.00000E-01	9.39368E-02	1.20300E-01	2.63632E-02
3.00000E-01	1.27979E-01	1.89677E-01	6.16977E-02
4.00000E-01	1.62192E-01	2.59389E-01	9.7197[illegible]E-02
5.00000E-01	1.98432E-01	3.25105E-01	1.26673E-[illegible]1
6.00000E-01	2.36287E-01	3.83043E-01	1.46756E-01
7.00000E-01	2.74046E-01	4.30394E-01	1.56348E-01
8.00000E-01	3.09504E-01	4.65637E-01	1.561[illegible]E-01
9.00000E-01	3.40566E-01	4.88616E-01	1.[illegible]049E-01
1.00000E+00	3.65658E-01	5.00342E-01	1.34684E-01

ε = 1.0000000000E-01

x_i	u_i	$u(x_i)$	$u(x_i) - u_i$
0.00000E+00	0.00000E+00	0.00000E+00	0.00000E+00
1.00000E-01	4.53669E-01	6.17630[illegible]01	1.63961E-01
2.00000E-01	6.81912E-01	8.06[illegible]5E-01	1.24863E-01
3.00000E-01	7.69407E-01	8.31381[illegible]E-01	6.19737E-02
4.00000E-01	7.83610E-01	8.0[illegible]1E-01	2.55111E-02
5.00000E-01	7.65471E-01	[illegible]6317E-01	9.84591E-03
6.00000E-01	7.35458E-01	7.39442E-01	3.98372E-03
7.00000E-01	7.02292E-01	[illegible]4114E-01	1.82230E-03
8.00000E-01	6.69194E-01	6.7[illegible]087E-01	8.93138E-04
9.00000E-01	6.37194E-01	6.37563E-01	3.68919E-04
1.00000E+00	6.06559E-01	6.06567E-01	8.08603E-06

ε = 1.0000000000E-02

x_i	u_i	$u(x_i)$	$u(x_i) - u_i$
0.00000E+00	0.00000E+00	0.00000E+00	0.00000E+00
1.00000E-01	[illegible]72830E-01	9.50136E-01	7.73055E-02
2.00000E-01	8.98991E-01	9.04779E-01	5.78868E-03
3.00000E-01	[illegible]59894E-01	8.60700E-01	8.06185E-04
4.00000E-01	8.18368E-01	8.18731E-01	3.62239E-04
5.00000E-01	7.78586E-01	7.78805E-01	2.19624E-04
6.00000E-01	7.40705E-01	7.40830E-01	1.25205E-04
7.00000E-01	7.04651E-01	7.04711E-01	6.00472E-05
8.00000E-01	6.70342E-01	6.70361E-01	1.91183E-05
9.00000E-01	6.37696E-01	6.37696E-01	3.74709E-07
1.00000E+00	6.06635E-01	6.06635E-01	0.00000E+00

Table 2.

h= 1.0000000000E-01

ε = 1.0000000000E-04

x_i	u_i	$u(x_i)$	$u(x_i) - u_i$
0.00000E+00	0.00000E+00	0.00000E+00	0.00000E+00
1.00000E-01	9.50701E-01	9.51229E-01	5.28090E-04
2.00000E-01	9.05089E-01	9.04837E-01	2.51481E-04
3.00000E-01	8.60924E-01	8.60708E-01	2.15695E-04
4.00000E-01	8.18918E-01	8.18733E-01	1.85730E-04
5.00000E-01	7.78965E-01	7.78806E-01	1.59425E-04
6.00000E-01	7.40964E-01	7.40830E-01	1.34301E-04
7.00000E-01	7.04819E-01	7.04711E-01	1.08938E-04
8.00000E-01	6.70439E-01	6.70361E-01	7.83732E-05
9.00000E-01	6.37739E-01	6.37696E-01	4.30147E-05
1.00000E+00	6.06635E-01	6.06635E-01	0.00000E+00

ε =1.0000000000E-05

x_i	u_i	$u(x_i)$	$u(x_i) - u_i$
0.00000E+00	0.00000E+00	0.00000E+00	0.00000E+00
1.00000E-01	9.51458E-01	9.51229E-01	2.28243E-04
2.00000E-01	9.05099E-01	9.04837E-01	2.61371E-04
3.00000E-01	8.60931E-01	8.60708E-01	2.22463E-04
4.00000E-01	8.18923E-01	8.18733E-01	1.90602E-04
5.00000E-01	7.78969E-01	7.78806E-01	1.62868E-04
6.00000E-01	7.40967E-01	7.40830E-01	1.36662E-04
7.00000E-01	7.04821E-01	7.04711E-01	1.09551E-04
8.00000E-01	6.70440E-01	6.70361E-01	7.92093E-05
9.00000E-01	6.37739E-01	6.37696E-01	4.34014E-05
1.00000E+00	6.06635E-01	6.06635E-01	0.00000E+00

ε =1.0000000000E-06

x_i	u_i	$u(x_i)$	$u(x_i) - u_i$
0.00000E+00	0.00000E+00	0.00000E+00	0.00000E+00
1.00000E-01	9.51533E-01	9.51229E-01	3.03928E-04
2.00000E-01	9.05100E-01	9.04837E-01	2.62309E-04
3.00000E-01	8.60932E-01	8.60708E-01	2.23140E-04
4.00000E-01	8.18924E-01	8.18733E-01	1.91089E-04
5.00000E-01	7.78969E-01	7.78806E-01	1.63213E-04
6.00000E-01	7.40967E-01	7.40830E-01	1.36898E-04
7.00000E-01	7.04821E-01	7.04711E-01	1.09704E-04
8.00000E-01	6.70440E-01	6.70361E-01	7.92979E-05
9.00000E-01	6.37739E-01	6.37696E-01	4.34402E-05
1.00000E+00	6.06635E-01	6.06635E-01	0.00000E+00

Table 3.

h= 1.0000000000E-01

ε = 1.0000000000E+00

x_i	u_i	$u(x_i)$	$u(x_i) - u_i$
0.00000E+00	0.00000E+00	0.00000E+00	0.00000E+00
1.00000E-01	1.22383E-01	5.56937E-02	6.66894E-02
2.00000E-01	2.15896E-01	1.20300E-01	9.55961E-02
3.00000E-01	2.88932E-01	1.89677E-01	9.92552E-02
4.00000E-01	3.46294E-01	2.59389E-01	8.69048E-02
5.00000E-01	3.90762E-01	3.25105E-01	6.56564E-02
6.00000E-01	4.24049E-01	3.83043E-01	4.10061E-02
7.00000E-01	4.47387E-01	4.30394E-01	1.69926E-02
8.00000E-01	4.61865E-01	4.65637E-01	3.771[illegible]E-03
9.00000E-01	4.68604E-01	4.88616E-01	2.00[illegible]E-02
1.00000E+00	4.68801E-01	5.00342E-01	3.154[illegible]E-02

ε = 1.0000000000E-01

x_i	u_i	$u(x_i)$	$u(x_i) - u_i$
0.00000E+00	0.00000E+00	0.00000E+00	0.00000E+00
1.00000E-01	6.42351E-01	6.17630E-01	2.47207E-02
2.00000E-01	8.23598E-01	8.06775E-01	1.68232E-02
3.00000E-01	8.39768E-01	8.31389E-01	8.38693E-03
4.00000E-01	8.11375E-01	8.09121E-01	2.25361E-03
5.00000E-01	7.74792E-01	7.75317E-01	5.24906E-04
6.00000E-01	7.38218E-01	7.39441E-01	1.22370E-03
7.00000E-01	7.03027E-01	7.04114E-01	1.08669E-03
8.00000E-01	6.69372E-01	6.70087E-01	7.14721E-04
9.00000E-01	6.37233E-01	6.37563E-01	3.29300E-04
1.00000E+00	6.06567E-01	6.06567E-01	7.54881E-11

ε = 1.0000000000E-02

x_i	u_i	$u(x_i)$	$u(x_i) - u_i$
0.00000E+00	0.00000E+00	0.00000E+00	0.00000E+00
1.00000E-01	9.49334E-01	9.50136E-01	8.01509E-04
2.00000E-01	9.03964E-01	9.04779E-01	7.15443E-04
3.00000E-01	8.60180E-01	8.60700E-01	5.20289E-04
4.00000E-01	8.18382E-01	8.18731E-01	3.48264E-04
5.00000E-01	7.78586E-01	7.78805E-01	2.19026E-04
6.00000E-01	7.40705E-01	7.40830E-01	1.25183E-04
7.00000E-01	7.04651E-01	7.04711E-01	6.00464E-05
8.00000E-01	6.70342E-01	6.70361E-01	1.91182E-05
9.00000E-01	6.37696E-01	6.37696E-01	3.74710E-07
1.00000E+00	6.06635E-01	6.06635E-01	0.00000E+00

Table 4.

h= 1.0000000000E-01

ε = 1.0000000000E-04

x_i	u_i	$u(x_i)$	$u(x_i) - u_i$
0.00000E+00	0.00000E+00	0.00000E+00	0.00000E+00
1.00000E-01	9.51528E-01	9.51229E-01	2.98094E-04
2.00000E-01	9.05090E-01	9.04837E-01	2.52054E-04
3.00000E-01	8.60924E-01	8.60708E-01	2.15695E-04
4.00000E-01	8.18918E-01	8.18733E-01	1.85730E-04
5.00000E-01	7.78965E-01	7.78806E-01	1.59423E-04
6.00000E-01	7.40964E-01	7.40830E-01	1.34301E-04
7.00000E-01	7.04819E-01	7.04711E-01	1.08023E-04
8.00000E-01	6.70439E-01	6.70361E-01	7.83232E-05
9.00000E-01	6.37739E-01	6.37696E-01	4.30137E-05
1.00000E+00	6.06635E-01	6.06635E-01	0.00000E+00

ε = 1.0000000000E-05

x_i	u_i	$u(x_i)$	$u(x_i) - u_i$
0.00000E+00	0.00000E+00	0.00000E+00	0.00000E+00
1.00000E-01	9.51540E-01	9.51229E-01	3.10913E-04
2.00000E-01	9.05099E-01	9.04837E-01	2.61376E-04
3.00000E-01	8.60931E-01	8.60708E-01	2.22463E-04
4.00000E-01	8.18923E-01	8.18733E-01	1.90602E-04
5.00000E-01	7.78969E-01	7.78806E-01	1.62868E-04
6.00000E-01	7.40967E-01	7.40830E-01	1.36662E-04
7.00000E-01	7.04821E-01	7.04711E-01	1.09551E-04
8.00000E-01	6.70440E-01	6.70361E-01	7.92093E-05
9.00000E-01	6.37739E-01	6.37696E-01	4.34014E-05
1.00000E+00	6.06635E-01	6.06635E-01	0.00000E+00

ε = 1.0000000000E-06

x_i	u_i	$u(x_i)$	$u(x_i) - u_i$
0.00000E+00	0.00000E+00	0.00000E+00	0.00000E+00
1.00000E-01	9.51542E-01	9.51229E-01	3.12195E-04
2.00000E-01	9.05102E-01	9.04837E-01	2.62309E-04
3.00000E-01	8.60932E-01	8.60708E-01	2.23140E-04
4.00000E-01	8.18924E-01	8.18733E-01	1.91089E-04
5.00000E-01	7.78969E-01	7.78806E-01	1.63213E-04
6.00000E-01	7.40967E-01	7.40830E-01	1.36898E-04
7.00000E-01	7.04821E-01	7.04711E-01	1.09704E-04
8.00000E-01	6.70440E-01	6.70361E-01	7.92979E-05
9.00000E-01	6.37739E-01	6.37696E-01	4.34402E-05
1.00000E+00	6.06635E-01	6.06635E-01	0.00000E+00