ISSN 2249 –5460

INTERNATIONAL JOURNAL OF MATHEMATICAL SCIENCES, TECHNOLOGY AND HUMANITIES

www.internationalejournals.com

Available online at www.internationaleJournals.com

International eJournals

International Journal of Mathematical Sciences, Technology and Humanities 110 (2014) 1177–1194

IDEALS IN PARTIALLY ORDERED TERNARY SEMIGROUPS

V. Siva Rami Reddy[1], V. Sambasiva Rao[2], A. Anjaneyulu [3],A. Gangadhara Rao [4].
Dept. of Mathematics,[1] NRI Engineering College, Guntur,
[2]Acharya Nagarjuna University, Guntur, [3,4]V S R & N V R College, Tenali.

[1]reddyvs13@gmail.com, [3]anjaneyulu.addala@gmail.com, [4]raoag1967@gmail.com

ABSTRACT

In this paper, the terms, partially ordered ternary semigroup, po ternary subsemigroup, po ternary subsemigroup generated by a subset, two sided identity of a po ternary semigroup, left zero, right zero, zero of a po ternary semigroup, po left ideal, po lateral ideal, po right ideal, po two sided ideal and po ideal in a po ternary semigroup are introduced. It is proved that, if T is a po ternary semigroup and A ⊆ T, B ⊆ T, then (i) A ⊆ (A], (ii) ((A]] = (A], (iii) (A](B](C] ⊆ (ABC] and (iv) A ⊆ B ⇒ A ⊆ (B], (v) A ⊆ B ⇒ (A] ⊆ (B]. It is proved that the nonempty intersection of any family of po ternary subsemigroups of a po ternary semigroup T is a po ternary subsemigroup of T. It is proved that (1) the nonempty intersection of any family of po left ideals (or po lateral ideals or po right ideals or po two sided ideals or po ideals) of a po ternary semigroup T is a po left ideal (or po lateral ideals or po right ideals or po two sided ideals or po ideals) of T, (2) the union of any family of po left ideals (or po lateral ideals or po right ideals or po two sided ideals or po ideals) of a po ternary semigroup T is a po left ideal (or po lateral ideals or po right ideals or po two sided ideals or po ideals) of T. Let T be a po-ternary semigroup and A is a nonempty subset of T, then it is proved that (1) L(A) = (A ∪ TTA], (2) M(A) = (A ∪ TAT∪ TTATT], (3) T(A) = (A ∪TTA∪ATT∪ TTATT] and (4) J(A) = (A ∪TTA∪TAT∪ATT∪ TTATT].

Mathematical subject classification (2010) : 20M07; 20M11; 20M12.

KEY WORDS : partially ordered ternary semigroup, po ternary subsemigroup, po ternary subsemigroup generated by a subset, cyclic po ternary subsemigroup of a po ternary semigroup, two sided identity of a po ternary semigroup, zero of a po ternary semigroup, po left ideal, po lateral ideal, po right ideal, two sided po ideal and po ideal.

1. INTRODUCTION:

The algebraic theory of semigroups was widely studied by CLIFFORD [2], [3], PETRICH [11] and LYAPIN [10]. The ideal theory in general semigroups was developed by ANJANEYULU [1]. The theory of ternary algebraic systems was introduced by LEHMER [8] in 1932. LEHMER investigated certain algebraic systems called triplexes which turn out to be commutative ternary groups. Ternary semigroups are universal algebras with one associative ternary operation. The notion of ternary semigroup was known to BANACH who is credited with example of a ternary semigroup which can not reduce to a semigroup. SANTIAGO [12] developed the theory of ternary semigroups. SIOSON [15] introduced the ideal theory in ternary semigroups. He also introduced the notion of regular ternary. In this paper we introduce the notions of ordered subsemigroups and characterize ordered subsemigroups and the notions of ordered ideals and characterize ordered ideals in po ternary semigroups.

1. PARTIALLY ORDERED TERNARY SEMIGROUPS :

DEFINITION 2.1 : A ternary semigroup T is said to be a ***partially ordered ternary semigroup*** if T is a partially ordered set such that $a \le b \Rightarrow [aa_1a_2] \le [ba_1a_2]$, $[a_1aa_2] \le [a_1ba_2]$, $[a_1a_2a] \le [a_1a_2b]$ for all $a, b, a_1, a_2 \in T$.

NOTE 2.2 : A partially ordered ternary semigroup is also called as po ternary semigroup or ordered ternary semigroup.

NOTATION 2.3 : Let T be a po ternary semigroup and S be a nonempty subset of T. If H is a nonempty subset of S, we denote $\{s \in S : s \le h$ for some $h \in H\}$ by $(H]_S$.

NOTATION 2.4 : Let T be a po ternary semigroup and S be a nonempty subset of T. If H is a nonempty subset of S, we denote $\{s \in S : h \le s$ for some $h \in H\}$ by $[H)_S$.

NOTE 2.5 : $(H]_T$ and $[H)_T$ are simply denoted by $(H]$ and $[H)$ respectively.

THEOREM 2.6 : Let T be a po-ternary semigroup and A ⊆ T, B ⊆ T. Then (i) A ⊆ (A], (ii) ((A]] = (A], (iii) (A](B](C] ⊆ (ABC] and (iv) A ⊆ B ⇒ A ⊆ (B], (v) A ⊆ B ⇒ (A] ⊆ (B].

***Proof* :** (i) Let $x \in$ A. $x \in$ A, $x \in$ T and $x \le x \Rightarrow x \in$ (A]. Therefore A $\subseteq$ (A].

(ii) Let $x \in$ ((A]] $\Rightarrow x \le y$ for some $y \in$ (A].

$y \in$ (A] $\Rightarrow y \le z$ for some $z \in$ A.

$x \le y,\ y \le z \Rightarrow x \le z$.

$x \le z, z \in$ A $\Rightarrow x \in$ (A].

Therefore ((A]] $\subseteq$ (A] and from (i) (A] = ((A]] and hence (A]] = (A].

(iii) Let $x \in$ (A] (B] (C] $\Rightarrow x = abc$ where $a \in$ (A], $b \in$ (B], $c \in$ (C].

$a \in$ (A] $\Rightarrow a \le \alpha$ for some $\alpha \in$ A $\Rightarrow abc \le \alpha bc$.

$b \in$ (B] $\Rightarrow b \le \beta$ for some $\beta \in$ B $\Rightarrow \alpha bc \le \alpha \beta c$.

$c \in$ (C] $\Rightarrow c \le \gamma$ for some $\gamma \in$ C $\Rightarrow \alpha \beta c \le \alpha \beta \gamma$.

Now $x = abc \le \alpha bc \le \alpha \beta c \le \alpha \beta \gamma$ where $\alpha \beta \gamma \in$ ABC $\Rightarrow x \in$ (ABC].

Therefore (A](B](C] $\subseteq$ (ABC].

(iv) From (i) B $\subseteq$ (B] $\Rightarrow$ A $\subseteq$ B $\subseteq$ (B].

(v) A $\subseteq$ B $\Rightarrow$ A $\subseteq$ (B] $\Rightarrow$ (A] $\subseteq$ ((B]] = (B].

Therefore (A] $\subseteq$ (B].

DEFINITION 2.7 : An element ***a*** of a po ternary semigroup T is said to be a ***left identity*** of T provided $aat = att = t$ and $t \le a$ for all $t \in$ T.

NOTE 2.8 : Left identity element a of a po ternary semigroup T is also called as ***left unital element.***

DEFINITION 2.9 : An element ***a*** of a po ternary semigroup T is said to be a ***right identity*** of T provided $taa = tta = t$ and $t \le a$ for all $t \in$ T.

NOTE 2.10 : Right identity element a of a po ternary semigroup T is also called as ***right unital element.***

DEFINITION 2.11 : An element '***a***' of a po ternary semigroup T is said to be a ***two sided identity*** provided it is both a left identity and a right identity of T.

NOTE 2.12 : An element '***a***' of a po ternary semigroup T is a ***two sided identity*** provided $aat = att = taa = tta = t$ and $t \leq a$ for all $t \in$ T.

NOTE 2.13 : Two-sided identity element of a po ternary semigroup T is also called as ***bi-unital element***.

THEOREM 2.14 : **If *a* is a left identity element and *b* is a right identity element of a po ternary semigroup T, then *a* = *b*.**

Proof: Since a is a left identity of T, $aat = att = t$ and $t \leq a$ for all $t \in$ T and
hence $aab = abb = b$ and $b \leq a$.
Since b is a right identity of T, $tbb = bbt = t$ and $t \leq b$ for all $t \in$ T and
hence $abb = bba = a$ and $a \leq b$.
Now $b \leq a$ and $a \leq b \Rightarrow a = b$.

THEOREM 2.15 : A po ternary semigroup T has at most one two sided identity.

***Proof* :** Let a, b be two sided identity elements of the po ternary semigroup T.
Now a can be considered as a left identity and b can be considered as a right identity of T.
By theorem 2.14, $a = b$. Then T has at most one two sided identity.

DEFINITION 2.16 : An element a of a po ternary semigroup T is said to be a ***left zero*** of T provided $abc = a$ and $a \leq b$, $a \leq c$ for all $b, c \in$ T.

DEFINITION 2.17 : An element a of a po ternary semigroup T is said to be a ***lateral zero*** of T provided $bac = a$ and $a \leq b$, $a \leq c$ for all $b, c \in$ T.

DEFINITION 2.18 : An element a of a po ternary semigroup T is said to be a ***right zero*** of T provided $bca = a$ and $a \leq b$, $a \leq c$ for all $b, c \in$ T.

DEFINITION 2.19 : An element a of a po ternary semigroup T is said to be a ***two sided zero*** of T provided $abc = bca = a$ and $a \leq b$, $a \leq c$ for all $b, c \in$ T.

NOTE 2.20 : If a is a two sided zero of a po ternary semigroup T, then a is both a left zero and a right zero of T.

DEFINITION 2.21 : An element a of a po ternary semigroup T is said to be a ***zero*** of T provided $abc = bac = bca = a$ and $a \leq b$, $a \leq c$ for all $b,c \in$ T.

NOTE 2.22 : If a is a zero of po ternary semigroup T, then a is a left zero, lateral zero and right zero of T.

THEOREM 2.23 : If *a* is a left zero, *b* is a lateral zero and *c* is a right zero of a po ternary semigroup T , then *a* = *b* = *c*.

Proof : Since a is a left zero of T, $abc = a$ and $a \leq b$, $a \leq c$ for all $b,c \in$ T.
Since b is a lateral zero of T, $abc = b$ and $b \leq a$, $b \leq c$ for all $a,c \in$ T.
Since c is a right zero of T, $abc = c$ and $c \leq a$, $c \leq b$ for all $a,b \in$ T.
Therefore $abc = a = b = c$.

THEOREM 2.24 : Any po ternary semigroup has at most one zero element.

Proof : Let a,b,c be three zeros of a po ternary semigroup T.
Now a can be considered as a left zero ,
b can be considered as a lateral zero and
c can be considered as a right zero of T.
By theorem 2.23, $a = b = c$.
Then T has at most one zero.

NOTE 2.25 : The zero (if exists) of a po ternary semigroup is usually denoted by 0.

2. PARTIALLY ORDERED TERNARY SUBSEMIGROUPS :

DEFINITION 3.1 : Let T be a po ternary semigroup. A nonempty subset S of T is said to be a ***po ternarysubsemigroup*** of T if (i) $abc \in$ S for all a, b, $c \in$ S, (ii) $t \in$ T, $s \in$ S, $t \leq s \Rightarrow t \in$ S.

NOTE 3.2 : A nonempty subset S of a po ternary semigroup T is a po ternary subsemigroup of T iff (1) SSS ⊆ S, (2) (S] = S.

EXAMPLE 3.3 : Let Z be the set of all integers. Define multiplication on Z by $[xyz] = \min\{x, y, z\}$ for all $x, y, z \in$ Z. Then Z is a po ternary semigroup. Let Z^- be the set of all negative integers. Then Z^- is a po ternary subsemigroup of Z.

EXAMPLE 3.4 : Let T = [0,1]. Then T is po ternary semigroup under the usual multiplication and usual order relation. Let S = [0, 1/2]. Then S is po ternary subsemigroup of T.

THEOREM 3.5 : The nonempty intersection of two po ternary subsemigroups of a po ternary semigroup T is a po ternary subsemigroup of T.

***Proof* :** Let S_1, S_2 be two po ternary subsemigroups of T. Let $a, b, c \in S_1 \cap S_2$.
$a, b, c \in S_1 \cap S_2 \Rightarrow a, b, c \in S_1$ and $a, b, c \in S_2$.
$a, b, c \in S_1$, S_1 is a po ternary subsemigroup of T $\Rightarrow abc \in S_1$ and $(S_1] = S_1$
$a, b, c \in S_2$, S_2 is a po ternary subsemigroup of T $\Rightarrow abc \in S_2$ and $(S_2] = S_2$
$abc \in S_1$, $abc \in S_2 \Rightarrow abc \in S_1 \cap S_2$ and $S_1 \cap S_2 \subseteq S_1$, $S_1 \cap S_2 \subseteq S_2$
$\Rightarrow (S_1 \cap S_2] \subseteq (S_1] = S_1$ and $(S_1 \cap S_2] \subseteq (S_2] = S_2 \Rightarrow (S_1 \cap S_2] \subseteq S_1 \cap S_2 \Rightarrow (S_1 \cap S_2] = S_1 \cap S_2$ and hence $S_1 \cap S_2$ is a po ternary subsemigroup of T.

THEOREM 3.6 : The nonempty intersection of any family of po ternary subsemigroups of a po ternary semigroup T is a po ternary subsemigroup of T.

***Proof* :** Let $\{S_\alpha\}_{\alpha \in \Delta}$ be a family of po ternary subsemigroups of T and let S = $\bigcap_{\alpha \in \Delta} S_\alpha$.

Let $a, b, c \in$ S.
$a, b, c \in S \Rightarrow a, b, c \in \bigcap_{\alpha \in \Delta} S_\alpha \Rightarrow a, b, c \in S_\alpha$ for all $\alpha \in \Delta$.
$a, b, c \in S_\alpha$, S_α is a po ternary subsemigroup of T $\Rightarrow abc \in S_\alpha$ and $(S_\alpha] = S_\alpha$

$\Rightarrow abc \in S_\alpha$ for all $\alpha \in \Delta \Rightarrow abc \in \bigcap_{\alpha\in\Delta} S_\alpha$ and $(\bigcap_{\alpha\in\Delta} S_\alpha] \subseteq \bigcap_{\alpha\in\Delta} S_\alpha$

$\Rightarrow abc \in$ S and (S] ⊆ S. Therefore S is a po ternary subsemigroup of T.

DEFINITION 3.7 : Let T be a po ternary semigroup and A be a nonempty subset of T. The smallest po ternary subsemigroup of T containing A is called a ***po ternary subsemigroup of* T *generated by* A**. It is denoted by (A).

THEOREM 3.8 : Let T be a po ternary semigroup and A be a nonempty subset of T. Then (A) = The intersection of all po ternary subsemigroups of T containing A.

***Proof* :** Let Δ be the set of all po ternary subsemigroups of T containing A.

Since T is a po ternary subsemigroup of T containing A, T ∈ Δ. So $\Delta \neq \emptyset$.

Let $S^* = \bigcap_{S\in\Delta} S$. Since $A \subseteq$ S for all S ∈ Δ, $A \subseteq S^*$ and hence $S^* \neq \emptyset$.

By theorem 3.6, S^* is a po ternary subsemigroup of T.

Since $S^* \subseteq$ S for all S ∈ Δ, S^* is the smallest po ternary subsemigroup of T containing A.

Therefore $S^* =$ (A).

3. PARTIALLY ORDERED TERNARY IDEALS :

DEFINITION 4.1 : A nonempty subset A of a po ternary semigroup T is said to be ***po left ternary ideal*** or ***po left ideal*** of T if i) $b, c \in$ T, $a \in$ A $\Rightarrow bca \in$ A ii) $a \in$ A and $t \in$ T such that $t \leq a \Rightarrow t \in$ A

NOTE 4.2 : A nonempty subset A of a po ternary semigroup T is a po left ternary ideal of T if and only if i) TTA ⊆ A ii) (A] ⊆ A.

DEFINITION 4.3 : A nonempty subset A of a po ternary semigroup T is said to be ***po lateral ternary ideal*** or ***po lateral ideal*** of T if i) $b, c \in$ T, $a \in$ A $\Rightarrow bac \in$ A ii) $a \in$ A and $t \in$ T such that $t \leq a \Rightarrow t \in$ A.

NOTE 4.4 : A nonempty subset A of a po ternary semigroup T is a po lateral ternary ideal of T if and only if i) TAT ∪ TTATT $\subseteq$ A ii) (A] $\subseteq$ A.

DEFINITION 4.5 : A nonempty subset A of a po ternary semigroup T is said to be ***po right ternary ideal*** or ***po right ideal*** of T if i) $b, c \in$ T, $a \in$ A $\Rightarrow$ $abc \in$ A ii) $a \in$ A and $t \in$ T such that $t \leq a \Rightarrow t \in$ A

NOTE 4.6 : A nonempty subset A of a po ternary semigroup T is a po right ternary ideal of T if and only if i) ATT $\subseteq$ A ii) (A] $\subseteq$ A.

DEFINITION 4.7 : A nonempty subset A of a po ternary semigroup T is said to be ***po two sided ternary ideal*** or ***po two sided ideal*** of T if i) $b, c \in$ T, $a \in$ A $\Rightarrow$ $bca \in$ A, $abc \in$ A ii) $a \in$ A and $t \in$ T such that $t \leq a \Rightarrow t \in$ A.

NOTE 4.8 : A nonempty subset A of a po ternary semigroup T is a po two sided ternary ideal of T if and only if i) TTA $\subseteq$ A, ATT $\subseteq$ A ii) (A] $\subseteq$ A.

NOTE 4.9 : A nonempty subset A of a po-ternary semigroup T is a po two sided ideal of T if and only if it is both a po left ideal and a po right ideal of T .

DEFINITION 4.10 : A nonempty subset A of a po ternary semigroup T is said to be ***po ternary ideal*** or ***po ideal*** of T if i) $b, c \in$ T, $a \in$ A $\Rightarrow$ $bca \in$ A, $bac \in$ A, $abc \in$ A ii) $a \in$ A and $t \in$ T such that $t \leq a \Rightarrow t \in$ A.

NOTE 4.11 : A nonempty subset A of a po ternary semigroup T is a po ternary ideal of T if and only if i) TTA $\subseteq$ A, TAT $\subseteq$ A, ATT $\subseteq$ A ii) (A] $\subseteq$ A.

NOTE 4.12 : A nonempty subset A of a po ternary semigroup T is a po ideal of T if and only if it is left po ideal, lateral po ideal and right po ideal of T.

EXAMPLE 4.13 : Let N be the set of all natural numbers. Define the ternary operation from $N \times N \times N \rightarrow$ N as (a, b, c) = $a.b.c$ where '.' is usual multiplication and ordered relation ≤ on N. Then N is a po ternary semigroup and A = 3 N is a po ideal of the po ternary semigroup N.

THEOREM 4.14 : The nonempty intersection of any two po left ideals (or po lateral ideals or po right ideals or po two sided ideals or po ideals) of a po ternary semigroup T is a po left ideal (or po lateral ideal or po right ideal or po two sided ideal or po ideal) of T.

Proof : Let A , B be two po left ideals of T.

Let $a \in A \bigcap B$ and $b, c \in T$

$a \in A \bigcap B \Rightarrow a \in A$ and $a \in B$

$a \in A$; $b, c \in T$, A is a po left ideal of T $\Rightarrow bca \in A$.

$a \in B$; $b, c \in T$, B is a po left ideal of T $\Rightarrow bca \in B$.

$bca \in A, bca \in B \Rightarrow bca \in A \bigcap B$.

Let $a \in A \bigcap B$ and $t \in T$ such that $t \leq a$.

$a \in A \bigcap B \Rightarrow a \in A$ and $a \in B$.

$a \in A, t \in T, t \leq a$, A is a po left ideal of T $\Rightarrow t \in A$.

$a \in B, t \in T, t \leq a$, B is a po left ideal of T $\Rightarrow t \in B$.

Therefore $t \in A, t \in B \Rightarrow t \in A \bigcap B$.

Hence $A \bigcap B$ is a po left ideal of T.

Similarly we can prove the other cases.

THEOREM 4.15 : The nonempty intersection of any family of po left ideals (or po lateral ideals or po right ideals or po two sided ideals or po ideals) of a po ternary semigroup T is a po left ideal (or po lateral ideal or po right ideal or po two sided ideal or po ideal) of T.

proof : Let $\{A_\alpha\}_{\alpha \in \Delta}$ be a family of po left ideals of T and let $A = \bigcap_{\alpha \in \Delta} A_\alpha$

Let $a \in A$; $b , c \in T$.

Now $a \in A \Rightarrow a \in \bigcap_{\alpha \in \Delta} A_\alpha \Rightarrow a \in A_\alpha$ for each $\alpha \in \Delta$.

$a \in A_\alpha$, $b , c \in T$, A_α is a po left ideal of T $\Rightarrow bca \in A_\alpha$.

$bca \in A_\alpha$ for all $\alpha \in \Delta \Rightarrow bca \in \bigcap_{\alpha\in\Delta} A_\alpha = A$.

Let $a \in$ A and $t \in$ T such that $t \le a$.

$a \in$ A = $\bigcap_{\alpha\in\Delta} A_\alpha \Rightarrow a \in A_\alpha$ for each $\alpha \in \Delta$.

$a \in A_\alpha$, $t \in$ T , $t \le a$, A_α is a po left ideal of T $\Rightarrow$ $t \in A_\alpha$ for each $\alpha \in \Delta$

$t \in A_\alpha$ for each $\alpha \in \Delta \Rightarrow t \in \bigcap_{\alpha\in\Delta} A_\alpha \Rightarrow t \in A$.

Hence A is a po left ideal of T.

Similarly we can prove the other cases.

THEOREM 4.16 : The union of any two po left ideals (or po lateral ideals or po right ideals or po two sided ideals or po ideals) of a po ternary semigroup T is a po left ideal (or po lateral ideal or po right ideal or po two sided ideal or po ideal) of T.

Proof : Let A_1, A_2 be two po left ideals of a po ternary semigroup T.

Let $A = A_1 \cup A_2$. Clearly A is a nonempty subset of T.

Let $a \in A$ and $b,c \in$ T. Now $a \in A \Rightarrow a \in A_1 \cup A_2 \Rightarrow a \in A_1$ or $a \in A_2$.

If $a \in A_1$ then $a \in A_1$; b , $c \in$ T; A_1 is a po left ideal of T $\Rightarrow bca \in A_1$

$bca \in A_1 \subseteq A_1 \cup A_2 = A \Rightarrow bca \in A$.

If $a \in A_2$ then $a \in A_2$; b , $c \in$ T; A_2 is a po left ideal of T $\Rightarrow bca \in A_2$

$bca \in A_2 \subseteq A_1 \cup A_2 = A \Rightarrow bca \in A$.

Let $a \in A$ and $t \in$ T such that $t \le a$.

$a \in A \Rightarrow a \in A_1 \cup A_2 \Rightarrow a \in A_1$ or $a \in A_2$.

If $a \in A_1$ then $a \in A_1$, $t \in$ T, $t \le a$, A_1 is a po left ideal of T $\Rightarrow t \in A_1 \subseteq A_1 \cup A_2 = A$

If $a \in A_2$ then $a \in A_2$, $t \in$ T, $t \le a$, A_2 is a po left ideal of T $\Rightarrow t \in A_2 \subseteq A_1 \cup A_2 = A$

Therefore $a \in$ A and $t \in$ T such that $t \leq a \Rightarrow t \in$ A.

Hence A is a po left ideal of T.

Similarly we can prove the other cases.

THEOREM 4.17 : The union of any family of po left ideals (or po lateral ideals or po right ideals or po two sided ideals or po ideals) of a po ternary semigroup T is a po left ideal (or po lateral ideal or po right ideal or po two sided ideal or po ideal) of T.

Proof : Let $\{A_\alpha\}_{\alpha\in\Delta}$ be a family of po left ideals of a po ternary semigroup T.

Let $A = \bigcup_{\alpha\in\Delta} A_\alpha$. Clearly A is a non-empty subset of T.

Let $a \in A$ and b , $c \in$T. $a \in A \Rightarrow a \in \bigcup_{\alpha\in\Delta} A_\alpha \Rightarrow a \in A_\alpha$ for some $\alpha \in \Delta$.

$a \in A_\alpha$, b , $c \in$T, A_α is a po left ideal of T $\Rightarrow bca \in A_\alpha \subseteq \bigcup_{\alpha\in\Delta} A_\alpha = A \Rightarrow bca \in A$.

Let $a \in$ A and $t \in$ T such that $t \leq a$.

$a \in A \Rightarrow a \in \bigcup_{\alpha\in\Delta} A_\alpha \Rightarrow a \in A_\alpha$ for some $\alpha \in \Delta$.

$a \in A_\alpha$, $t \in$ T , $t \leq a$, A_α is a po left ideal of T $\Rightarrow a \in A_\alpha$ for some $\alpha \in \Delta \Rightarrow t \in A_\alpha \subseteq$

$\bigcup_{\alpha\in\Delta} A_\alpha$ Therefore $t \in \bigcup_{\alpha\in\Delta} A_\alpha$ = A

Therefore A is a po left ideal of T.

Similarly we can prove the other cases.

4. <u>TERNARY IDEALS GENERATED BY A SUBSET</u>:

DEFINITION 5.1 : Let T be a po ternary semigroup and A be a nonempty subset of T. The smallest po left ideal of T containing A is called ***po left ideal of* T *generated by A*** and it is denoted by L(A).

THEOREM 5.2 : If T is a po ternary semigroup and A is a nonempty subset of T, then L(A) = (A ∪ TTA].

Proof **:** Let b, c ∈ T and r ∈ (A ∪ TTA].

r ∈ (A ∪ TTA] ⇒ $r \le x$ for some x ∈ A ∪ TTA.

x ∈ A ∪ TTA ⇒ x ∈ A or x ∈ TTA.

If x ∈ A then $bcr \le bcx$ ∈ TTA ⊆ A ∪ TTA ⇒ bcr ∈ (A ∪ TTA].
If x ∈ TTA then $x = yza$ where y, z ∈ T and a ∈ A.
$bcr \le bcx = bcyza$ ∈ TTA ⊆ A ∪ TTA ⇒ bcr ∈ (A ∪ TTA].

Therefore b, c ∈ T and r ∈ (A ∪ TTA] ⇒ bcr ∈ (A ∪ TTA].
Let s ∈ (A ∪ TTA] and t ∈ T such that $t \le s$.
s ∈ (A ∪ TTA] ⇒ $s \le x$ for some x ∈ A ∪ TTA.
$t \le s$, $s \le x \Rightarrow t \le x$. t ∈ T, $t \le x$, x ∈ A ∪ TTA ⇒ t ∈ (A ∪ TTA].
Therefore (A ∪ TTA] is a po left ideal of T.

Let L be a po left ideal of T containing A. A ⊆ L, L is a po left ideal of T ⇒ TTA ⊆ TTL ⊆ L.
A ⊆ L, TTA ⊆ L ⇒ A ∪ TTA ⊆ L ⇒ (A ∪ TTA] ⊆ L

Therefore (A ∪ TTA] is the smallest po left ideal containing A.
Therefore L(A) = (A ∪ TTA].

NOTE 5.3 : (A ∪ TTA] is also denoted as (T^1T^1A]

THEOREM 5.4 : The po left ideal (or po lateral ideal or po right ideal or po two sided ideal or po ideal) of a po ternary semigroup T generated by a nonempty subset A is the intersection of all po left ideals ideal (or po lateral ideals or po right ideals or po two sided ideals or po ideals) of T containing A.

Proof **:** Let Δ be the set of all po left ideals of T containing A.

Since T itself is a po left ideal of T containing A, T ∈ Δ. So Δ ≠ ∅.

Let $T^* = \bigcap_{T \in \Delta} T$. Since $A \subseteq T$ for all T ∈ Δ, it follows that $A \subseteq T^*$.

By theorem 4.15, T^* is a po left ideal of T.

Let K be a po left ideal of T containing A.

Clearly A ⊆ K and K is a po left ideal of T.

Therefore K ∈ Δ ⇒T^* ⊆ K. Therefore T^* is the po left ideal of T generated by A.

DEFINITION 5.5 : A po left ideal A of a po ternary semigroup T is said to be the ***principal po left ideal generated by an element a*** if A is a po left ideal generated by $\{a\}$ for some a ∈ T. It is denoted by L(a).

THEOREM 5.6 : If T is a po ternary semigroup and *a* ∈ T then L(*a*) = (*a* ∪ TT*a*].

***Proof* :** Proof is similar to theorem 5.2.

DEFINITION 5.7 : Let T be a po ternary semigroup and A be a nonempty subset of T. The smallest po lateral ideal of T containing A is called ***po lateral ideal of* T *generated by A*** and it is denoted by M (A).

THEOREM 5.8 : If T is a po ternary semigroup and A is a nonempty subset of T, then M(A) = (A ∪ TAT∪ TTATT].

***Proof* :** Let b, c ∈ T and r ∈ (A ∪ TTA∪ TTATT].

r ∈ (A ∪ TTA∪ TTATT] ⇒ $r \leq x$ for some x ∈ A ∪ TTA∪ TTATT.

x ∈ A ∪ TTA∪ TTATT ⇒ x ∈ A or x ∈ TTA or x ∈ TTATT.

If x ∈ A then $bcr \leq bcx$ ∈ TTA ⊆ A ∪ TTA∪ TTATT ⇒ bcr ∈ (A ∪ TTA∪ TTATT].
If x ∈ TTA then $x = yza$ where y, z ∈ T and a ∈ A.
$bcr \leq bcx = bcyza$ ∈ TTA ⊆ A ∪ TTA∪ TTATT ⇒ bcr ∈ (A ∪ TTA∪ TTATT].
If x ∈ TTATT then $x = yzauv$ where y, z, u, v ∈ T and a ∈ A.
$bcr \leq bcx = bcyzauv$ ∈ TTATT ⊆ A ∪ TTA∪ TTATT ⇒ bcr ∈ (A ∪ TTA∪ TTATT].

Therefore b, c ∈ T and r ∈ (A ∪ TTA∪ TTATT] ⇒ bcr ∈ (A ∪ TTA∪ TTATT].
Let s ∈ (A ∪ TTA∪ TTATT] and t ∈ T such that $t \leq s$.

$s \in$ (A ∪ TTA∪ TTATT] ⇒ $s \le x$ for some $x \in$ A ∪ TTA∪ TTATT.

$t \le s$, $s \le x \Rightarrow t \le x$.

$t \in$ T, $t \le x$, $x \in$ A ∪ TTA∪ TTATT ⇒ $t \in$ (A ∪ TTA∪ TTATT].

Therefore (A ∪ TTA∪ TTATT] is a po lateral ideal of T.

Let M be a po lateral ideal of T containing A.

A ⊆ M, M is a po lateral ideal of T ⇒ TTA ⊆ TTM ⊆ M.

A ⊆ M, M is a po lateral ideal of T ⇒ TTATT ⊆ TTMTT ⊆ M.

A ⊆ M, TTA ⊆ M, TTATT⊆ M ⇒ A ∪ TTA∪TTATT ⊆ M ⇒ (A ∪ TTA∪ TTATT] ⊆ M.

Therefore (A ∪ TTA∪ TTATT] is the smallest po lateral ideal containing A.

Therefore M(A) = (A ∪ TTA∪ TTATT].

DEFINITION 5.9 : A po lateral ideal A of a po ternary semigroup T is said to be the ***principal po lateral ideal generated by a*** if A is a po lateral ideal generated by $\{a\}$ for some $a \in$ T. It is denoted by M (a).

THEOREM 5.10 : If T is a po ternary semigroup and a ∈ T then M(a) = (a ∪ TaT∪ TTaTT].

***Proof* :** Proof is similar to theorem 5.8.

DEFINITION 5.11 : Let T be a po ternary semigroup and A be a nonempty subset of T. The smallest po right ideal of T containing A is called ***po right ideal of* T *generated by A*** and it is denoted by R(A).

THEOREM 5.12 : Let T be a po ternary semigroup and A is a nonempty subset of T, then R(A) = (A ∪ ATT].

***Proof* :** Proof is similar to theorem 5.2.

DEFINITION 5.13 : A po right ideal A of a po ternary semigroup T is said to be the ***principal po right ideal generated by an element a*** if A is a po right ideal generated by $\{a\}$ for some $a \in$ T. It is denoted by R(a).

THEOREM 5.14 : If T is a po ternary semigroup and *a* ∈ T then R(*a*) = (*a* ∪ *a*TT].

***Proof* :** Proof is similar to theorem 5.2.

DEFINITION 5.15 : Let T be a po ternary semigroup and A be a nonempty subset of T. The smallest po two sided ideal of T containing A is called ***po two sided ideal of* T *generated by A*** and it is denoted by T(A).

THEOREM 5.16 : Let T be a po ternary semigroup and A is a nonempty subset of T, then T_1 (A) = (A ∪TTA∪ATT∪ TTATT].

***Proof* :** Let $b, c \in$ T and $r \in$ (A ∪ TTA∪ATT ∪TTATT].

$r \in$ (A ∪ TTA∪ATT ∪ TTATT] ⇒ $r \leq x$ for some $x \in$ A ∪ TTA∪ATT ∪ TTATT.

$x \in$ A ∪ TTA∪ATT ∪ TTATT ⇒ $x \in$ A or $x \in$ TTA or $x \in$ ATT or $x \in$ TTATT.

If $x \in$ A then $bcr \leq bcx \in$ TTA ⊆ A ∪ TTA∪ATT ∪ TTATT
⇒ $bcr \in$ (A ∪ TTA∪ATT ∪ TTATT].
If $x \in$ TTA then $x = yza$ where $y, z \in$ T and $a \in$ A.
$bcr \leq bcx = bcyza \in$ TTA ⊆ A ∪ TTA∪ATT ∪ TTATT ⇒ $bcr \in$ (A ∪ TTA∪ATT ∪ TTATT].
If $x \in$ ATT then $x = ayz$ where $y, z \in$ T and $a \in$ A.
$bcr \leq bcx = bcayz \in$ TTATT ⊆ A ∪ TTA∪ATT ∪ TTATT ⇒ $bcr \in$ (A ∪ TTA∪ATT ∪ TTATT].
If $x \in$ TTATT then $x = yzauv$ where $y, z, u, v \in$ T and $a \in$ A.
$bcr \leq bcx = bcyzauv \in$ TTATT ⊆ A ∪ TTA∪ATT ∪ TTATT
⇒ $bcr \in$ (A ∪ TTA∪ATT ∪TATT].
Therefore $b, c \in$ T and $r \in$ (A ∪ TTA∪ATT ∪ TTATT] ⇒ $bcr \in$ (A ∪ TTA∪ATT ∪ TTATT].
Let $s \in$ (A ∪ TTA∪ATT ∪ TTATT] and $t \in$ T such that $t \leq s$.
$s \in$ (A ∪ TTA∪ATT ∪ TTATT] ⇒ $s \leq x$ for some $x \in$ A ∪ TTA∪ATT ∪ TTATT.
$t \leq s, s \leq x \Rightarrow t \leq x$.
$t \in$ T, $t \leq x$, $x \in$ A ∪ TTA∪ATT ∪ TTATT ⇒ $t \in$ (A ∪ TTA∪ATT ∪ TTATT].
Therefore (A ∪ TTA∪ATT ∪ TTATT] is a po two sided ideal of T.
Let T_1 be a po two sided ideal of T containing A.

$A \subseteq T_1$, T_1 is a po two sided ideal of T ⇒ $TTA \subseteq TT T_1 \subseteq T_1$.

$A \subseteq T_1$, T_1 is a po two sided ideal of T ⇒ $ATT \subseteq T_1TT \subseteq T_1$.

$A \subseteq T_1$, T_1 is a po two sided ideal of T ⇒ $TTATT \subseteq TT T_1TT \subseteq T_1$.

$A \subseteq T_1$, $TTA \subseteq T_1$, $ATT \subseteq T_1$, $TTATT \subseteq T_1 \Rightarrow A \cup TTA \cup ATT \cup TTATT \subseteq T_1$

$\Rightarrow (A \cup TTA \cup ATT \cup TTATT] \subseteq T_1$.

Therefore $(A \cup TTA \cup ATT \cup TTATT]$ is the smallest po lateral ideal containing A.

Therefore $T_1(A) = (A \cup TTA \cup ATT \cup TTATT]$.

DEFINITION 5.17 : A po two sided ideal A of a po ternary semigroup T is said to be the ***principal po two sided ideal generated by an element a*** if A is a po two sided ideal generated by $\{a\}$ for some $a \in T$. It is denoted by T(a).

THEOREM 5.18 : If T is a po ternary semigroup and $a \in T$ then $T(a) = (a \cup TTa \cup aTT \cup TTaTT]$.

Proof **:** Proof is similar to theorem 5.16.

DEFINITION 5.19 : An ideal A of a po ternary semigroup T is said to be a ***proper po ideal*** of T if A is different from T.

DEFINITION 5.20 : An ideal A of a po ternary semigroup T is said to be a ***trivial po ideal*** provided T\ A is singleton.

DEFINITION 5.21 : Let T be a po ternary semigroup and A be a nonempty subset of T. The smallest po ideal of T containing A is called ***po ideal of* T *generated by A*** and it is denoted by J(A).

THEOREM 5.22 : Let T be a po ternary semigroup and A is a nonempty subset of T, then $J(A) = (A \cup TTA \cup TAT \cup ATT \cup TTATT]$.

Proof **:** Let $b, c \in T$ and $r \in (A \cup TTA \cup TAT \cup ATT \cup TTATT]$.

$r \in (A \cup TTA \cup TAT \cup ATT \cup TTATT] \Rightarrow r \leq x$ for some $x \in A \cup TTA \cup TAT \cup ATT \cup TTATT$.

$x \in A \cup TTA \cup TAT \cup ATT \cup TTATT \Rightarrow x \in A$ or $x \in TTA$ or $x \in TAT$ or $x \in ATT$ or $x \in TTATT$.

If $x \in A$ then $bcr \leq bcx \in TTA \subseteq A \cup TTA \cup TAT \cup ATT \cup TTATT$

$\Rightarrow bcr \in$ (A ∪ TTA∪TAT∪ATT∪ TTATT].

If $x \in$ TTA then $x = yza$ where $y, z \in$ T and $a \in$ A.

$bcr \leq bcx = bcyza \in$ TTA ⊆ A ∪ TTA∪TAT∪ATT∪ TTATT

$\Rightarrow bcr \in$ (A ∪ TTA∪TAT∪ATT∪ TTATT].

If $x \in$ TAT then $x = yaz$ where $y, z \in$ T and $a \in$ A.

$bcr \leq bcx = bcyaz \in$ TAT ⊆ A ∪ TTA∪TAT∪ATT∪ TTATT

$\Rightarrow bcr \in$ (A ∪ TTA∪TAT∪ATT∪ TTATT].

If $x \in$ ATT then $x = ayz$ where $y, z \in$ T and $a \in$ A.

$bcr \leq bcx = bcayz \in$ TTATT ⊆ A ∪ TTA∪TAT∪ATT∪ TTATT

$\Rightarrow bcr \in$ (A ∪ TTA∪TAT∪ATT∪ TTATT].

If $x \in$ TTATT then $x = yzauv$ where $y, z, u, v \in$ T and $a \in$ A.

$bcr \leq bcx = bcyzauv \in$ TTATT ⊆ A ∪ TTA∪TAT∪ATT∪ TTATT

$\Rightarrow bcr \in$ (A ∪ TTA∪TAT∪ATT∪ TTATT].

Therefore $b, c \in$ T and $r \in$ (A ∪ TTA∪TAT∪ATT∪ TTATT]

$\Rightarrow bcr \in$ (A ∪ TTA∪TAT∪ATT∪ TTATT].

Let $s \in$ (A ∪ TTA∪TAT∪ATT∪ TTATT] and $t \in$ T such that $t \leq s$.

$s \in$ (A ∪ TTA∪TAT∪ATT∪ TTATT] $\Rightarrow s \leq x$ for some $x \in$ A ∪ TTA∪TAT∪ATT∪ TTATT.

$t \leq s, s \leq x \Rightarrow t \leq x$.

$t \in$ T, $t \leq x$, $x \in$ A ∪ TTA∪TAT∪ATT∪ TTATT $\Rightarrow$ $t \in$ (A ∪ TTA∪TAT∪ATT∪ TTATT].

Therefore (A ∪ TTA∪TAT∪ATT∪ TTATT] is a po ideal of T.

Let J be a po ideal of T containing A.

A ⊆ J, J is a po ideal of T ⇒ TTA ⊆ TT J ⊆ J.

A ⊆ J, J is a po ideal of T ⇒ TAT ⊆ TJT ⊆ J.

A ⊆ J, J is a po ideal of T ⇒ ATT ⊆ JTT ⊆ J.

A ⊆ J, J is a po ideal of T ⇒ TTATT ⊆ TT JTT ⊆ J.

A ⊆ J, TTA ⊆ J, TAT ⊆ J, ATT ⊆ J, TTATT ⊆ J ⇒ A ∪ TTA∪TAT∪ATT∪ TTATT ⊆ J

⇒ (A ∪ TTA∪TAT∪ATT∪ TTATT] ⊆ J.

Therefore (A ∪ TTA∪TAT∪ATT∪ TTATT] is the smallest po ideal containing A.

Therefore J(A) = (A ∪ TTA∪TAT∪ATT∪ TTATT].

DEFINITION 5.23 : A po ideal A of a po ternary semigroup T is said to be the ***principal po ideal generated by an element a*** if A is a po-ideal generated by $\{a\}$ for some $a \in$ T. It is denoted by J(a) or < a >.

THEOREM 5.24 : If T is a po ternary semigroup and $a \in$ T then J(a) = ($a \cup TTa \cup TaT \cup aTT \cup TTaTT$].

Proof **:** Proof is similar to theorem 5.22.

References

[1] **Anjaneyulu. A .**, Structure and ideal theory of Duo semigroups, Semigroup forum, vol .22(1981),237-276.

[2] **Clifford A.H and Preston G.B**., The algebroic theory of semigroups , vol – I American Math.Society,Province(1961).

[3] **Clifford A.H and Preston G.B**., The algebroic theory of semigroups , vol – II American Math.Society,Province(1967).

[4] **Hewitt. E. and Zuckerman H.S**., Ternary opertions and semigroups, semigroups, Proc. Sympos.WayneStateUniv.,Detroit,1968,33-83.

[5] **Iampan . A**., Lateral ideals of ternary semigroups , Ukrainian Math, Bull., 4 (2007), 323-334.

[6] **Kar.S** ., On ideals in ternary semigroups . Int. J. Math. Gen. Sci., 18 (2003) 3013- 3023.

[7] **Kar.S and Maity.B.K**., *Some ideals of ternary semigroups* . Analele Stintifice Ale Universitath "ALI CUZA" DIN IASI(S.N) Mathematica, Tumul LVII. 2011-12.

[8] **Lehmer . D.H**., A ternary analave of abelian groups , Amer. J. Math., 39 (1932), 329-338.

[9] **Los. J.,** ***On the extending of models I,*** Fundamenta Math. 42(1955), 38 – 54.

[10] **Lyapin.E.S**., Realisation of ternary semigroup, Russian Modern Algebra, Leningrad University,Leningrad,1981,pp.43-48.

[11] **Petrch.M**., Introduction to semigroups , Merril publishing company, Columbus, Ohio (1973).

[12] **Santiago. M. L**. and Bala S.S., " Ternary semigroups" Semigroups Forum,Vol. 81,no. 2, pp. 380-388,2010.

[13] **Sarala. Y, Anjaneyulu. A** and **Madhusudhana Rao. D.,** *On ternary semigroups*, International eJournal of Mathematics,Engineering and Technology accepted for publication.

[14] **Sarala. Y, Anjaneyulu. A** and **Madhusudhana Rao. D.,** *Ideals in ternary semigroups*, submitted for publication in International eJournal of Mathematics, Engineering and Technology acceptedforpublication.

[15] **Sioson. F. M.,** ***Ideal theory in ternary semigroups,*** Math. Japon.10(1965), 63-84.

* * * * *