ISSN 2249 –5460

Available online at www.internationaleJournals.com

INTERNATIONAL JOURNAL OF MATHEMATICAL SCIENCES, TECHNOLOGY AND HUMANITIES

www.internationalejournals.com

International eJournals

International Journal of Mathematical Sciences, Technology and Humanities 125 (2014) 1349 - 1354

Proof of Non-existence of odd Perfect Numbers

P. Vivekanand

Maths IIT and Olympiad Faculty Raman High School, Subedari, Hanamkonda, Warangal District. E-mail: vivekputchala@gmail.com

ABSTRACT:

An odd perfect number is a number whose sum of factors is equal to double the number it self and the number is odd. The existence or non- existence is not known since the time of Euclid. With some basic results found in internet the author proves the theorem. The results used from internet are mentioned in the proof. It has been 2000 years since the problem is posed and unsolved. So let us go into the proof. We have used Egyptian fractions and had obtained only a single page proof. It stresses the importance of Egyptian fractions in number theory.

INTRODUCTION:

The problem states that there is no odd perfect number. A perfect number is a number whose sum of factors is double itself. For example 6 has factors 1,2,3,6 who sum is 12 which equal to twice 6. But clearly 6 is even number. Our problem is to prove that an odd perfect number does not exist.

The previous results obtained are due to great mathematicians like Euler, Euclid who proved that odd perfect number is of the form 4K + 1 and number of factors of it should be 2n where n is odd. After this many mathematicians gave boundaries for perfect numbers for example from Wolfram Math World the latest research has been given as we see below. To obtain these results mathematicians used lot of Jargon but good results which could not prove the non existence of odd perfect numbers. Here the author gives a simple proof which could be under stood by an elementary student of number theory. The Wolfram Math World gives the following results on odd perfect numbers, if they exist.

In Book IX of *The Elements,* Euclid gave a method for constructing perfect numbers (Dickson 2005, p. 3), although this method applies only to *even* perfect numbers. In a 1638 letter to Mersenne, Descartes proposed that every even perfect number is of Euclid's form, and stated that he saw no reason why an odd perfect number could not exist (Dickson 2005, p. 12). Descartes was therefore among the first to consider the existence of odd perfect numbers; prior to Descartes, many authors had implicitly assumed (without proof) that the perfect numbers generated by Euclid's construction comprised all possible perfect numbers (Dickson 2005, pp. 6-12). In 1657, Frenicle repeated Descartes' belief that every even perfect number is of Euclid's form and that there was no reason odd perfect number could not exist. Like Frenicle, Euler also considered odd perfect numbers.

To this day, it is not known if any odd perfect numbers exist, although numbers up to 10^{300} have been checked without success, making the existence of odd perfect numbers appear unlikely (Brent *et al.* 1991; Guy 1994, p. 44). The following table summarizes the development of ever-higher bounds for the smallest possible odd perfect number. There is a project underway at http://www.oddperfect.org/ seeking to extend the limit beyond 10^{300}.

author	bound
Kanold (1957)	10^{20}
Tuckerman (1973)	10^{36}
Hagis (1973)	10^{50}
Brent and Cohen (1989)	10^{160}
Brent et al. (1991)	10^{300}

Euler showed that an odd perfect number, if it exists, must be of the form

$$N = p^{4\lambda+1} Q^2, \qquad (1)$$

where p is a prime of the form $4n+1$ (Fermat's 4*n*+1 theorem; Burton 1989), a result similar to that derived by Frenicle in 1657 (Dickson 2005, pp. 14 and 19). In other words, an odd perfect number must be of the form

$$N = p^{\alpha} q_1^{2\beta_1} \cdots q_r^{2\beta_r} \qquad (2)$$

for distinct odd primes p, q_1, ..., q_r with $p \equiv \alpha \equiv 1$ (mod 4). Steuerwald (1937) subsequently proved that the β_is cannot all be 1 (Yamada 2005).

Touchard (1953) proved that an odd perfect number, if it exists, must be of the form $12k+1$ or $36k+9$(Holdener 2002).

In 1896, Stuyvaert stated that an odd perfect number must be a sum of two squares (Dickson 2005, p. 28). In 1887, Sylvester conjectured and in 1925, Gradshtein proved that any odd perfect number must have at least six distinct prime factors (Ball and Coxeter 1987). Hagis (1980) showed that odd perfect numbers must have at least eight distinct prime factors, in which case, the number is divisible by 15 (Voight 2003).

In 1888, Catalan proved that if an odd perfect number is not divisible by 3, 5, or 7, it has at least 26 distinct prime aliquot factors, and this was extended to 27 by Norton (1960). Norton (1960) showed that odd perfect numbers not divisible by 3 or 5, it must have at least 15 distinct prime factors. Neilsen (2006), improving the bound of Hagis (1980), showed that if an odd perfect number is not divisible by 3, it must have at least 12 distinct prime factors. Nielsen (2006) also showed that a general odd perfect number, if it exists, must have at least 9 distinct prime factors.

More recently, Hare (2005) has shown that any odd perfect number must have 75 or more prime factors. Improving this bound requires the factorization of several large numbers (Hare), and attempts are currently underway to perform these factorizations using the elliptic curve factorization method at mersenneforum.org and OddPerfect.org.

For the *largest* prime factor of an odd perfect number, Iannucci (1999, 2000) and Jenkins (2003) have worked to find lower bounds. The largest three factors must be at least 100000007, 10007, and 101. Goto and Ohno (2006) verified that the largest factor must be at least 100000007 using an extension to the methods of Jenkins.

For the *smallest* prime factor of an odd perfect number with all even powers lower than six, Yamada (2005) determined an upper bound of $\exp\left(4.97401\times 10^{10}\right)$

For any odd perfect number with $r$prime factors and $1 \le i \le 5$, Kishore (1981) has established upper bounds for small factors of odd perfect numbers by showing that

$$p_i < 2^{2^{i-i}}\ (i). \qquad (3)$$

FORMULATION OF THE PROBLEM:

The odd perfect number is a number such that sum of all the factors of is double it self i.e. σ N= 2N where N belongs to odd numbers. With the help of computers and algorithms it was found by Brent et al. (1991) that an odd perfect number, if it exists is more than 10^{300}. Though Descartes was of the opinion odd perfect numbers can exists, our investigation in to large numbers show the results to be on the negative. Taking this as inspiration and its mystery around more than 2000 years the author proves here the non- existence of odd perfect numbers using method of contradiction.

FUTURE DIRECTIONS:

This proof can be adopted to know the existence of odd multi perfect number or not which could lead to understanding of primes as primes are involved in it.

PROOF:

Let 'N' be odd perfect Number, then

- It has 2n factor where n $\in$ odd ------------------------------ (1)
- The factors are of the form $4K_i+1$ or $4K_r - 1$ as they are all odd -------------------- (2)
- N is of the form 4K + 1 --------------------------(3)

  $\therefore$ $(4K_1 + 1)(4K_2 + 1) = 4K + 1 = N$

  $(4K_3 + 1)(4K_4 + 1) = N$

  .

  .

  .

  $(4K_r - 1)(4K_{r+1} - 1) = N$

- $\therefore$ It is clear that there are even no of factors of the form $4K_i + 1$ & even no of factors as the form $4K_r - 1$. as they occur in pairs.------------------------ (4)
- Also the sum of the reciprocals of factors of 'N' = 2 & if we subtract the factor '1' which is of the form $4K_i + 1$ from the sum of the reciprocals of all the factors, them the resultant sum = '1' now let no-unity factors of 'N' be d_1, d_2, d_3……d_{odd1}, S_1, S_2……S_{even1} where $d_i \in 4K_i + 1$ & $S_r \in 4K_r - 1$. now d_i are odd in number as '1' is removed from them [See result 4].

i.e. $\left[\frac{1}{d_1}+\frac{1}{d_2}........+\frac{1}{d_{ood1}}\right]+\left[\frac{1}{S_1}+\frac{1}{S_2}........+\frac{1}{S_{even1}}\right]=1$-------------------(5)

the above (1), (2), (3), (5)

are internet results, whose proofs are elementary.

Now consider the progression

$$\frac{1}{3}-\frac{1}{3^2}+\frac{1}{3^3}-\frac{1}{3^4}......+\frac{1}{3^{odd1}}-\frac{1}{2.3^{odd1}}=\frac{1}{2}$$ ----------------- (6)

and

$$\frac{1}{5}+\frac{1}{5^2}......+\frac{1}{5^{even1}}+\frac{1}{4}+\frac{1}{4.5^{even1}}=\frac{1}{2}$$ ------------------(7)

Now

Eq (5) – [eq (6) + eq (7)] when

Simplified

$$\Rightarrow \quad \frac{2Q_1}{odd_2}+\frac{1}{2.3^{odd1}}+\frac{4Q_2}{odd_3}-\frac{1}{4}\left[\frac{5^{even1}+1}{5^{even1}}\right]=0$$

$$\Rightarrow \quad \frac{2Q_1}{odd_2}+\frac{4Q_2}{odd_3}+\frac{1}{2.3^{odd1}}-\frac{1}{2}\left[\frac{4K_3+1}{4K_4+1}\right]=0$$

$$\Rightarrow \quad \frac{2Q_1}{odd_2}+\frac{4Q_2}{odd_3}+\frac{1}{2[4K_5-1]}-\frac{1}{2}\left[\frac{4K_3+1}{4K_4+1}\right]=0$$

$$\Rightarrow \quad \frac{2Q_1}{odd_2}+\frac{4Q_2}{odd_3}+\frac{1}{2}\left[\frac{4K_4+1+4K_6+1}{4K_7-1}\right]=0$$

$$\Rightarrow \quad \frac{2Q_1}{odd_2}+\frac{4Q_2}{odd_3}+\frac{1}{2}\left[\frac{4K_8+2}{4K_7-1}\right]=0$$

$$\Rightarrow \quad \frac{2Q_1}{odd_2}+\frac{4Q_2}{odd_3}+\frac{odd_5}{odd_6}=0 \qquad \Rightarrow \quad \frac{odd}{odd}=0 \quad \Rightarrow \mathrm{odd}=0$$

This is impossible

- The above proof is for case (i), where some factors are of the form $4K_i + 1$ and others of the form $4K_r – 1$.

- If we check carefully for the factors S_1, S_2………………..S_r mentioned in (5) where $S_r = 4K_r – 1$ can be changed to $S_r = 4K_r + 1$ without loss of generality even in this change the equations of the proof remain same without disturbing the structure of proof [Note all d_i,, S_r are not equal for odd perfect number N].
- For case (iii) when all factors are of the form $4K_r – 1$ cannot be true as the factor N it self is of the form 4K + 1.

  Hence it is proved that odd perfect number cannot exist for all cases (i), (ii), (iii)

CONCLUSION:

Therefore we conclude that odd perfect numbers do not exist. As we started with the existence of odd perfect numbers and we are getting contradiction. Therefore there are no odd perfect numbers.

BIBLIOGRAPHY:

1) Introduction to elementary number theory by Niven and Zuckerman.
2) Internet sources like Wolfram Math World etc.