INTERNATIONAL JOURNAL OF MATHEMATICAL SCIENCES, TECHNOLOGY AND HUMANITIES

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International eJournals

International Journal of Mathematical Sciences, Technology and Humanities 42 (2012) 422 – 439

Fixed Point Theorems for Weak Commutating Mapping

B.Nageswara Rao
Department of Basic Science and Humanities (Mathematics),
Costal Institute of Technology and Management, Narapam, Kothavalasa-535183,
Vijayanagaram district, A.P., India

Abstract: In this paper we have tried to introduce the concept of weak commutativity, weak commutativity and weak commutativity in 2-metric space. After this we have done some fixed point theorems for weak commutating mappings in 2-metric space.

Key Words: weak commutating mappings, commutating mappings

Introduction: Let T_1 and T_2 are two mappings from a metric space (X,) into itself, T_1 and T_2 are said to be commutative_ if $T_1 T_2 x = T_2 T_1 x$, for all x. for all x in X. Sessa [6] introduced the concept of weak commutatively in metric space. In subsequent years the condition of week commutatively was again made more weaker and weak_commutative was introduced in metric space. In recent years weak commutativity_ has been introduced and some theorems have been done for existence of fixed point for weak commutative maps in metric space such as [1], [4], [5] etc.

In this paper we have tried to introduce the concept of weak commutativity, weak commutativity and weak commutativity in 2-metric space. After this we have done some fixed point theorems for weak commutating mappings in 2-metric space. Some of our results generalizes the result of Kubaik [3], Hadzic [2] etc. and some of them are extensions of the results of metric spaces to 2-metric spaces such as [1], [4], [5] etc.

PRELIMINARIES: Here first we define weak commutativity, weak commutativity and weak commutativity and then we give an example to show that weak commutating mapping may not be a commutating mappings IN 2-metric space. Some of our results generalizes the result of Kubaik [3], Hadzic [2] etc., and some of them are extensions of the results of metric spaces to 2-metric spaces such as [2], [4], [5] etc.

1.**Premiminaries**: Here first we define week** commutativity, week* commutativity, and weak commuttivity and then we give an example to show that weak** commutating mapping may not be a commutating mapping.

(1.1) **DEFINITION**: Two self maps A and S of a 2-metric space (X,d) are called weak** commutating if A (X) $\subset$ S (X) and

$$d\,(A^2S^2x, S^2A^2x, a) \le d\,(A^2Sx, SA^2x, a) \le d\,(AS^2x, S^2Ax, a)$$
$$\le d\,(ASx, SAx, a) \le d\,(S^2x, A^2x, a)$$

for all x, a ε X.

(1.2) **DEFINITION**: If we retain only last inequality in (5.1.1), that is

$d\,(A^2S^2x, S^2A^2x, a) \le d\,(S^2x, A^2x, a)$ the definition reduces to weak* commutativity.

(1.3) **DEFINITION**: If we consider A and S as idempotent mappings i.e. A^2= A and S^2 = S, our definition reduces to weakly commutating pair of mappings {S.A.}.

1.1 **DEFINITION:** Two self maps A and B of a 2 – metric space (x,d) are called weak commutating if A(X) $\subset$ S(X) and $D(A^2S^2 x, S^2 A^2 x,a) \le d(A^2 Sx, SA^2 x,a) \le d(AS^2 x, S^2 Ax, a) \le d\,(ASx, Sax, a) \le d(S^2 x, A^2 x, a)$.
For all x,a ε X.

1.2 **DEFINITION:** If we retain only last ineuquality in (5.1.1), that is $d(A^2S^2 x, S^2 A^2 x,a) \le d(S^2 x, A^2 x, a)$ the definition reduces to weak commutativity.

1.3 **DEFINITION**: If we consider A and S as idempotent mappings i.e A^2 = A and S^2 = S, our definition reduces to weakly commutating pair of mappings {S,A}
i.e., $d(ASx, Sax,a) \le d\,(ASx, Sax, a)$

1.4 **EXAMPLE**: Let X = [0,1] with the 2-metric d defined as d(x,y,z) = min { | x-y | , | y-z [, | z – x | }.
Let A,B, S,T be defined as

$Ax = \frac{x}{x+4}$, $Bx = \frac{x}{x+6}$, $Sx = \frac{x}{2}$ and $Tx = \frac{x}{3}$, for all x ε X,

Then $A(X) = \left[0, \frac{1}{5}\right] \subset \left[0, \frac{1}{2}\right] = S(X)$ and

$D((A^2S^2 x, S^2 A^2 x,a) = \min \{ A^2S^2 x, - S^2 A^2 x| , | S^2 A^2 x - a| , | a – A^2 S^2 x| \}$

$= |\ A^2S^2 x, S^2 A^2 x\ | = \left| \frac{x}{5x+64} - \frac{x}{20x+64} \right|$

$= \frac{15x^2}{(5x+64)\,(20x+64)}$

$\leq \frac{15x^2}{15x+96)\,(10x+32)} = \frac{x}{5x+32} - \frac{x}{10x+32}$

$= d(A^2\, Sx, SA^2\, x, a)$

$D(A^2\, Sx, SA^2\, x, a) = \frac{5x^2}{(5x+32)\,(10x+32)} = \frac{(5/4)\, x^2}{((5/2)x+16)\,(5x+16)}$

$\leq \frac{3x^2}{(x+16)(4x+16)} = \frac{x}{x+16} - \frac{x}{4x+16} = d(AS^2\, x, S^2\, Ax, a)$

$D(AS^2\, x, S^2\, Ax, a) = \min\{\, |\ AS^2\, x - S^2\, Ax|, |S^2\, Ax - a|, |\ a - AS^2\, x|\ \}$

$= |\ AS^2\, x - S^2\, Ax|$

$= \frac{3x^2}{(x+16)\,(4x+16)}$

$\leq \frac{x^2}{\left(\frac{2}{3}x + \frac{32}{x}\right)(2x+8)} \leq \frac{x^2}{(x+8)\,(2x+8)}$

$= \left(\frac{x}{8+x}\right) - \left(\frac{x}{2x+8}\right) = d(ASx, DAX, a)$

D(ASx, SAx,a) = $\frac{x^2}{(x+8)\,(2x+8)} \frac{5x^2}{(x+8)\,(10x+40)}$

$\leq \frac{5x^2 + 12x}{4(5x+16)} = \frac{x}{4} - \frac{x}{5x+16} = d(S^2\, x, A^2\, x, a)$

Hence was conclude that

$D(AS^2\, x, S^2\, Ax, a) \leq D(A^2 Sx, SA^2\, x, a) \leq D(AS^2 x, S^2\, Ax, a)$

$\leq D(ASx, SAx, a) \leq D(S^2 x, A^2 x, a)$

For all x ε X and every a in X. But for any $x \neq O$, We have ASx = $\frac{x}{x+8}$ > $\frac{x}{2x+8}$ = SAx. //

(1.5) DEFINITION:

Let (X,d) be a 2-metric space and let T and I be mappings of X into itself. The map T is called rotative with respect to I if $d(Tx, I^2 x, a) \leq d(Ix, T^2x, a)$ for all x in X and every a in X. If T and I are idempotent maps, then the definition is obvious.

1.6. **THEOREM**: Let A, B, S and T be four self mappings of a complete 2-metric space (x,d) with d continuous such that $A^2, B^2 : X \to S^2(x) \cap T^2(X)$ and satisfy:

(I) $d(A^2 x, B^2 y, a) \leq c \max, \{ d(S^2 x, T^2 y,a), d(S^2 x, A^2 x, a)\ d(T^2 y, B^2 y, a), \frac{1}{2}[d(S^2 x, B^2 y, a) + d(T^2 y, A^2 x, e)] \}$

For all x,y,a in X, where U≤ O ≤ 1. If one of A,B,S and T is continuous and if A and B weak commutative with S and T respectively, then A,B,S and T have a unique common fixed point.

Proof: Let X_o be an arbitrary point of X and since $A^2(X)$ and $B^2(X)$ are contained in $S^2(X) \cap T^2(X)$, we can define sequence $\{X_n\}$ in X such that
$S^2x_{2n-1} = B^2 x_{2n-2}$ and $T^2x_{2n} = A^2x_{2n-1}$, for n = 1,2,3 ……………….. By (I) we have

$$
\begin{aligned}
D(S^2x_{2n-1}, T^2x_{2n}, a) &= d(B^2 x_{2n-2}, A^2x_{2n-1}, a) \\
&\leq c \max \{ d(S^2x_{2n-1}, T^2x_{2n-2}, a), \\
&\quad D(S^2x_{2n-1,} A^2x_{2n-1},a), \\
&\quad D(T^2x_{2n-2}, B^2x_{2n-2,} a), \\
&\quad \frac{1}{2} [d(S^2 x_{2n-1,} B^2 x_{2n-2,} a) + d (I^2 x_{2n-2}, A^2x_{2n-1}, a)]\} \\
&= c \max [d(S^2 x_{2n-1} : T^2 x_{2n-2}, a) , d(S^2 x_{2n-1}, T^2x_{2n}, a), \\
&\quad \frac{1}{2} d(T^2 x_{2n-2}, T^2 x_{2n}, a)\}
\end{aligned}
$$

Thus, $d (S^2 x_{2n-1}, T^2 x_{2n}, a) \leq c\ d(S^2 x_{2n-1}, T^2 x_{2n,} a)$ for n = 1,2 ……… and all a ε X. By induction we obtain

$d(S^2 x_{2n-1}, T^2 x_{2n},a) \leq c^{2n-1} d(S^2 X_1,T^2 x_o,a)$ …………………. (2)
and $d(S^2 x_{2n+1}, T^2 x_{2n,}a) \leq O^{2n-1} d(S^2 x_1, T^2 x_2, a)$ ……………… (3)

for n = 1,2 ……………………….. and all a ε X.

$$
\begin{aligned}
\text{Thus, } d(S^2 x_{2n-1}, S^2 x_{2n+1}, a) &\leq c^{2n-1} [d(S^2 x_1, T^2 x_o, a) \\
&\quad + d(S^2 x_1, T^2 x_2, a)] \\
&\leq (1 +C)^{2n-1} d(S^2 x_1, T^2 x_O, a) \quad \text{………(4)}
\end{aligned}
$$

Now for any n < m, it follows from triangle inequality and (4) that

$$D(5^2 x_{2n-1}, S^2 x_{2n-1}, a) \leq \frac{c^{2n-1}}{1-c} [d(S^2 x_1, T^2 x_O, a) + d(S^2 x_1, T^2 x_O, S^2 x_{2n-1})]$$

Further from triangle inequality and (4)

$$d(d(S^2 x_1, T^2 x_o, S^2 x_{2n-1}) \leq \sum_{i=1}^{m-1} d(S^2 x_{2i-1}, S^2 x_{2i-1}, Tx_O) + \sum_{i=1}^{n-1} d(S^2 x_{2i-1}, S^2 x_{2i-1}, S^2 x_1) = O$$

So that $\{S^2 x_{2n-1}\}$ is a Cauchy sequence, thus convergent to a point u in X.

Since, $d(T^2 x_{2n}, u, a) \leq d(T^2 x_{2n}, u, S^2 x_{2n-1}) + d(T^2 x_{2n}, S^2 x_{2n+1}, a) + d(S^2 x_{2n-1}, u, a)$

$$\leq c^{2n-1} d(S^2 x_1, T^2 x_O, u) + d(S^2 x_{2n-1}, u, a).$$

When $n \to \infty$, $d(T^2 x_{2n}, u, a) = O$ i.e, $\underset{n\to\infty}{Lt} T^2 x_{2n} = u$.

Thus, $\underset{n\to\infty}{Lt} S^2 x_{2n-1} = \underset{n\to\infty}{Lt} B^2 x_{2n-2} = \underset{n\to\infty}{Lt} T^2 x_{2n} = \underset{n\to\infty}{Lt} A^2 x_{2n-1} = u$.

Now suppose that …. Is continuous. We have the sequence $\{A=2 Sx_{2n}\}$ converges to Su. A being weak commutating with S, we deduce that $\{A^2 Sk_{2n}\}$ converge to Su. Now using the fact that $\{S^3 x_{2n+1}\}$ converges to Su and using condition (1) we have

$$d(A^2 Sx_{2n}, B^2 x_{2n+1}, a) \leq c \max, \{d(S^3 x_{2n}, T^2 x_{2n+1}, a), d(S^3 x_{2n}, A^2 Sx_{2n}, a), d(T^2 x_{2n+1}, B^2 x_{2n+1}, a), \frac{1}{2}[d(S^3 x_{2n}, B^2 x_{2n+1}, a) + d\{T^2 x_{2n+1}, A^2 Sx_{2n}, a)]\}.$$

When $n \to \infty$, we have Su = u and so $S^2 u = u$.

Now, $d(A^2 u, B^2 x_{2n+1}, a) \leq c \max, (d(S^2 u, T^2 x_{2n+1}, a)$
$d(S^2 u, A^2 u, a), d(T^2 x_{2n+1}, B^2 x_{2n+1}, a),$

$$\frac{1}{2}\ [d(S^2u, B^2\ x_{2n+1}, a)\ + d(T^2\ x_{2n+1}, A^2\ u, a)]\}$$

Letting $n \to \infty$, we have $d(A^2 u, u, a) \le c\ d(A^2 u, u, a) < d(A^2 u, u, a)$, a contradiction and therefore $A^2 u = u = S^2 u$.

Since the range of T^2 contains the range of A^2, let u_1 be a point in X such that $T^2 u_1 = u$. Then using (1) we have

$$\begin{aligned} d(u, B^2 u, a) &= d(A^2 u, B^2 u_1, a) \\ &\le c \max. \{ d(S^2 u, T^2 u_1, a), d(S^2 u, A^2 u, a), d(T^2 u_1, B^2 u_1, a), \\ &\quad \frac{1}{2} [d(S^2u, B^2\ u_1, a) + \ d(T^2 u_1, A^2 u, a)]\} \end{aligned}$$

i.e, $d(u, B^2 u_1, a) \le c\ d\ (u, B^2 u_1, a.)$ which implies that $u = B^2 u_1$. Since B is a weak commutating with T, we have $B^2 T^2 u_1 = T^2 B^2 u_1$ which implies that $B^2 u = T^2 u$. Using again (I)

$$\begin{aligned} d(u, B^2 u, a) &= d(A^2 u, B^2 u, a) \\ &\le c \max\ d(u, B^2 u_1, a),\ d(u,u,a)\ d(B^2 u, B^2 u, a) , \\ &\quad \frac{1}{2} [d(u, B^2 u, a) , + d (B^2 u, u, a)] \} \\ &= c\ d{*}u, B^2 u, a) \text{ which gives } d(u,B^2 u, a) = O \end{aligned}$$

Thus we get, $B^2 u = T^2 u = u$. Since A weak commutes with S we have $S^2 Au = AS^2 u$ and $S^2 Au = A^3 u = Au$. Now,

$$\begin{aligned} d(Au, u,a) &= d(A^3 u, B^2 u, a) \\ &\le c\ \max. \{ d(S^2 Au, T^2 u, a) , d(S^2 Au, A^3 u, a), d(T^2 u, B^2 u, a) \\ &\quad \frac{1}{2} [d(S^2 Au, B^2 u, a) + d(T^2 u, A^3 u, a)]\} \\ &= c\ \ d(Au, u, a) < a\ d(Au, u, a) \end{aligned}$$

Which implies that $Au = u$. Therefore $Au = Su = u$. Using (I), symmetric property of 2 – metric and week commutativity of {B,T} we can deduce that $Tu = Bu = u$.

Therefore u is a common fixed point of A, B, C and T. Analogously we can prove assuming continuity of T in place of S.

Now we suppose that A is continuous. Then the sequence $\{AS^2x_{2n}\}$ converges to Au. Since A weak commute with S, we have
$d(S^2 Ax_{2n}, Au, a) \le d(S^2 X_{2n}, A^2 x_{2n} + Au) + d(S^2 x_{2n} , A^2 x_{2n}, a)$

$+ d(AS^2 x_{2n} , Au, a)$

Which implies when $n \to \infty$, that the sequence $\{S^2 Ax_{2n}\}$ converges to Au. Using (I) and using the fact that $\{A^3 x_{2n}\}$ converges also to Au, Also

$$d(A^3x_{2n}, B^2 x_{2n+1}, a) = d(A^2 Ax_{2n}, B^2 x_{2n+1}, a)$$
$$\leq c \max, \{d(AS^2 x_{2n}, T^2x_{2n+1}, a),$$
$$d(AS^2 x_{2n}, AA^2 x_{2n,} a),$$
$$d(T^2 x_{2n+1}, B^2 x_{2n+1}, a),$$
$$\infty \ \frac{1}{2} [d(AS^2 x_{2n}, B^2 x_{2n+1}, a) + d(T^2 x_{2n+1}, AA^2 2n, a]$$

When $n \to \infty$. we have

$$d(Au, u,a) \leq c \max. \{d(Au,u,a) \ \frac{1}{2} [d(Au,u,a) + d(u,Au,a)]\}$$
$$= c\, d(Au,u,a), \text{ giving } Au = u \text{ and so } A^2 u = u.$$

As above we can show that $T^2 u = S^2 u = u$.
Since the range of S^2 contains the range of A^2, let u_2 be a point in X such that $S^2 u_2 = u$, Using (1) we have $d(A^2 u_2, u,a) = d(A^2 u_2, B^2 u, a)$

$$\leq c \max (d(u, u, a), d(u,A^2 u_2, a), d(u, u, a),$$
$$\frac{1}{2} [d(u,u,a) + d(u, A^2 u_2, a)$$

Which implies $A^2 u_2 = u$. Since X weak** commutating with S, we have $D(A^2 S^2u_2, S^2A^2 u_2,a) < d(S^2 u_2, A^2 u_2, a) = d(u, u, a) = O$ and therefore $S^2 u = S^2 A^2 u_2 = A^2 S^2 u_2 = A^2 u = u$

Now, $d(Su, u,a) = d(A^2 Su, S^2 u, a)$

$$\leq c \max. (d(S^3 u, B^2 u, a), d(S^3 u, A^2 Su, a), d(T^2 u, B^2 u, a),$$
$$\frac{1}{2} [d(S^3 u, B^2 u, a) + d(T^2 u, A^2 Su, a)]\}$$

$= c\, d(Su, u, a) < d(Su, u, a)$, which implies Su = u, and so, Su = Au = u. Since $B^2 Tu = TB^2 u$ and $T^2 Bu = BT^2 u$ and so $B^2 Tu = T^3 = Tu$ and $T^2 Bu = B^3 u = Bu$.

Now, $d(u,Tu,a) = d(A^2 u, B^2 Tu, a)$

$$\leq c \max. \{ d(u, Tu, a), d(u,u,a), d(Tu, Tu,a),$$
$$\frac{1}{2} [d(u, Tu,a) + d(Tu, u, a)]\}$$

$= c\, d(Tu, u,a) < d(Tu,u,a)$, which implies that Tu = u. Similarly we can prove that Bu = u and Tu = Bu = u. We have thus proved that Au = Bu = Su = Tu = u, i.e., u is the common fixed point of A,B,S and T. If the mapping B is continuous, instead of A, then analogously we can show that x is the common fixed point of A.B,S and T.

Now we claim that U is the unique common fixed point of A,B,S and T.

For this let $u^* \neq u$ be another fixed point, them such that $A(u^*) = Bu^* = u^*$, them

$d(u,u^*,a) = d(A^2 u, B^2 u^*,a)$

$\leq c \max, \{ d(u, u^*,a) , d(u,u,a), d(u^*, u^*,a) ,$
$\frac{1}{2} [d(u, u^*,a) + d(u^*, u,a)]\}$
$= c\, d(u, u^*, a) < d(u, u^*, a)$ which implies that $u = u^*$.

So, we proved that u is the unique common fixed point of A,B,S and T. //

(1.7) **COROLLARY:** Let $S,T : X \rightarrow x$ and either S or T be continuous. Then S and T have a common fixed point x if there exists two self mappings A,B of X and A (resp. B) weakly commutes with S(Resp.T). Further x is the unique common fixed point of A,B, S and T.

Proof: As A(resp. B) weakly commutes with S (resp. T). But weakly commutatively implies weak** commutativity. Thus the proof of Theorm [10] work.

Remark:
(1) The corollary (1.7) generalixes Theorem 1 of Kubaik [33] where continuity of both S and T and commutativity of both A and B with S and T are assumed, But assumptions in corollary (1.7) are much weaker than that of Kubaik [33] and thus theorem (1.6) is more general than Kubaik [33].

(1.8) **THEORM**: Let A, B, S and T be four self mappings of a complete 2 – metric space (X,d) such that (i) $A^2 (x) \subset T^2(x)$ and $S^2(X) \subseteq S^2 (X)$,

(ii) $d(A^2 x, B^2 y, a) \leq c \max. \{d(S^2 x, T^2 y, a), d(S^2 x, A^2 x, a), d(T^2 y, B^2 y, a),$
$\frac{1}{2} [d(S^2 x, B^2 y, a) + d(T^2 x, A^2 y, a)]\}$

For all x,y,a in X, where $O < c < 1$. If one of A,B, S and T is continuous and if A and B weak** commute with S and T respectively, then A,B, S and T have a unique common fixed point.

Proof: Similar to the proof of Theorem (5.1.6).
(1.9) <u>THEOREM:</u> Let S,T and I be three self mappings of complete 2-metric space (X,d)

with d continuous such that for all x,y,a in X either

$$d(S^2 x, T^2 u. a) \leq \frac{d(I^2 x, S^2 x, a)\, d(I^2 y, T^2 y, a) + \beta\, d(I^2 x, T^2 y, a) + d(I^2 y, S^2 x, a)}{d(T^2 x, S^2 x, a) + d(T^2 y, T^2 y.a)} \quad \text{--- } 1$$

If $d(T^2 x, S^2 x, a) + d(I^2 y, T^2 y, a) \neq O$

Where $1 < a < 2$ and $b \geq O$ or,

$d(S^2 x, T^2 y, a) = O$ if $d(T^2 x, S^2 x,a) + d(I^2 y, T^2 y, a) = o$ -- 2
Suppose that the range of I^2 contains the range of S^2 and T^2. If either

(X_1) I^2 is contionous, I is weak** commutating with S and T is rotative with respect to I or,
(A_2) I^2 is continuous, I is weak** commutating with T and S is rotative with respect to I or,
(A_3) S^2 is continuous, S is weak** commutating with I and T is rotative with respect to S or,
(A_3) S^2 is continuous, S is weak** commutating with I and T is retative with respect to S or,
(A_4) T^2 is continuous, T is weak** commutating with I and S is rotative with respect to T.

Then S,T and I have a unique common fixed point x. Further x is the unique common fixed point of S and I and T and I.

Proof: Let x_o be an arbitrary point in X, Since the range of I^2 contains the range of S^2. Let x_1 be a point in X such that $S^2 x_o = I^2 x_1$. Since the range of I^2 contains the range T^2 $x_1 = T^2 x_2$. In general

$T^2 x_{2n+1} = I^2 x_{2n-1}$, $S^2 x_{2n} = I^2 x_{2n+1}$ for n = 1,2 …………..
Put $d_{2n-1} = d(T^2 x_{2n-1}, S^2 x_{2n}, a)$ and $d_{2n} = d(S^2 x_{2n}, T^2 x_{2n+1}, a)$

For n = 1,2 ………………… Now we distinguish the three cases:-

(i) Let $d_{2n-1} \neq O$ and $d_{2n} \neq O$ for n = 1,2 ……………., we then have, $d_{2n-1} + d_{2n} = d(T^2 x_{2n+1}, a) \neq O$ for n = 1,2 …………….. Using inequality(i), we then have
$d_{2n} = d(S^2 x_{2n}, T^2 x_{2n+1}, a)$
$\leq$

$$a \; \frac{d(T^2 x_{2n-1,} \, S^2 x_{2n}, a)\, d(S^2 x_{2n+1}, a) + \beta\, d(S^2 x_{2n+1}, T^2 x_{2n+1}, a\, d(S^2 x_{2n}, S^2 x_{2n}, a)}{d(T^2 x_{2n-1}, G^2 x_{2n}, a) + d(S^2 x_{2n}, T^2 x_{2n+1}, a)} = a\frac{d_{2n-1} + d_{2n}}{d_{2n-1} + d_{2n}}$$

Then $\frac{d_{2n}}{d_{2n}} \leq a \; \frac{d_{2n-1}}{d_{2n-1} + d_{2n}} \Rightarrow d_{2n-1} + d_{2n} \leq a\, d_{2n-1} \Rightarrow d_{2n} \leq a\, d_{2n-1} - d_{2n} = (a-1)\, d_{2n-1} = c\, d_{2n-1}$

So, $d(S^2 x_{2n}, T^2 x_{2n+1}, a) = d_{2n} \leq cd_{2n-1} = cd\,(T^2 x_{2n-1}, S^2 x_{2n}, a)$ ………. (3)

For n = 1,2 ………., where c = (a-1), Similarly it can be proved that $d(T^2 x_{2n-1}, S^2 x_{2n}$

,a) = $d_{2n-1} \le c\, d_{2n-2} = c\, d(S^2 x_{2n-1}, T^2 x_{2n-1}, a)$ for n = 1,2 …….. and since $O < c < 1$. It follows that the sequence
$\{ S^2x_o, T^2 x_1, S^2x_2 \ldots\ldots, T^2 x_{2n-1}, S^2 x_{2n}, T^2 x_{2n+1} \ldots\ldots \}$ ……..(4)

Is a Cauchy sequence in the complete 2-metric space X and so has a limit u in X. Hence the sequence $\{S^2 x_{2n}\} = \{I^2 x_{2n+1}\}$ and $\{T^2 x_{2n-1}\} = \{T^2 x_{2n}\}$ converge to the point u because they are subsequence of the sequence (4).

Suppose first of all that I^2 is continuous, then the sequence $\{I^4 x_{2n}\}$ and $\{I^2 S^2 x_{2n}\}$ converge to point I^2 u. If I weak** commutes with S, we have $d(S^2 I^2 x_{2n}, I^2 u, a) \le d(S^2 I^2 x_{2n}, I^2 u, I^2 S^2 x_{2n}) + d(S^2 I^2 x_{2n}, I^2 S^2 x_{2n}, a) + d(I^2 S^2 x_{2n}, I^2 u, a) \le d(S^2 x_{2n}, I^2 x_{2n}, I^2 x_{2n}, I^2 u) + d(S^2 x_{2n}, I^2 x_{2n}, a) + d(I^2 S^2 x_{2n}, I^2 u, a)$ which implies on letting n tends to infinity that the sequence $\{S^2 I^2 x_{2n}\}$ also converges to I^2 u. Now we claim that $T^2 u = I^2 u$. Supposed not, then we have $d(I^2 u, T^2 u, a) > O$ and using inequality (1) we obtain

$$a\, d(S^2 I^2 x_{2n}, T^2 u, a) \le \frac{a\, d(I^4 X_{2n}, S^2 I^2 x_{2n}, a)\, d(I^2 u, T^2 u, a) + \beta\, d(I^4 x_{2n}, T^2 u, a)\, d(I^2 u, S^2 I^2 x_{2n}, a)}{d(I^4 x_{2n}, S^2 I^2 x_{2n}, a) + d(I^2 u, T^2 u, a)}$$

Letting $n \to \infty$, we deduce that $c(I^2 u, T^2 u, a) \le O$, a contradiction, Now suppose that $S^2 u \ne T^2 u$, then

$$d(S^2 u, T^2 u, a) \le (a+\beta)\ \frac{d(I^2 u, S^2 u, a) + d(I^2 u, T^2 u, a)}{d(I^2 u, S^2 u, a) + d(I^2 u, T^2 u, a)} = O$$

a contradiction, Thus $I^2 u = S^2 u = T^2 u$.

A similar conclusion is obtained if I is weak** commute with T. Let us suppose that S^2 is continuous instead of I^2. Then the sequence $\{S^4 x_{2n}\}$ and $\{S^2 I^2 x_{2n}\}$ converge to a point S^2 u. Since S weak** commute with I, We have the sequencey $\{I^2 S^2 x_{2n}\}$ also converges to S^2 u. Since the range of I^2 contains the range of S^2, there exists a point u_1 such that $I^2 u_1 = S^2 u_1$. Then if $T^2 u_1 \ne S^2 u$, we have

$$d(S^4 x_{2n}, T^2 u, a) \le a\ \frac{d(I^2 S^2 x_{2n}, S^2 S^2 x_{2n}, a) + d(I^2 u, T^2 u_1, a) + \beta\, d(I^2 S^2 x_{2b}, T^2 u_1, a) + d(T^2 u_1, S^2 S^2 x_{2n}, a)}{d(I^2 S^2 x_{2n}, S^2 S^2 x_{2n}, a) + d(I^2 u_1, T^2 u_1, a)}$$

when $n \to \infty$ we have

$$d(S^2 u, T^2 u_1, a) \le \frac{\beta\, d(I^2 u, T^2 u_1, a) + d(I^2 u_1, S^2 u, a)}{d(T^2 u_1, T^2 u_1, a)}$$

which implies that $d(S^2 u, T^2 u_1, a) \le O$, a contradiction.

Thus $S^2 u = T^2 u_1$. Now suppose that $S^2 u_1 \ne T^2 u_1$, then we have

$$D(S^2 u_1, T^2 u_1, a) \leq (a+\beta) \frac{d(I^2u_1, S^2u_1, a)\, d(I^2u_1, T^2u_1, a)}{d(I\, u_1, S\, u_1, a) + d(I\, u_1, T^2 u_1, a)} = 0,$$

a contradiction and so $I^2 u_1 = S^2 u_1 = T^2 u_1$.

A similar conclusions is achieved if one assumes that T^2 is continuous and T is weak** commutating with I.

(ii) Let $d_{2n-1} = O$ for some n. Then $I^2 x_{2n} = T^2 x2n_{-1} = S^2 x_{2n}$. We claim that $I^2 x_{2n} = T^2 x_{2n}$. Since otherwise if $d(I^2 x_{2n}, T^2 x_{2n}, a) > O$

$$O < d(I^2 x_{2n}, T^2 x_{2n}, a) = d(S^2 x_{2n}, T^2 x_{2n}, a) \leq \frac{d(I^2x_{2n}, S^2x_{2n}, a) + d(I^2x_{2n,} T^2x_{2n,} a) + \beta\, d(I^2x_{2n}, T^2x_{2n}, a)\, d(I^2x_{2n}, S^2x_{2n,} a}{d(I^2x_{2n}, S^2x_{2n}, a) + d(I^2x_{2n}, T^2x_{2n}, a)} = O$$

a contradiction. Thus, $I^2 x_{2n} = S^2 x_{2n} = T^2 x_{2n.}$

(iii) Let $d_{2n} = O$ for some n. Then $I^2 x_{2n+1} = S^2 x_{2n} = T^2 x_{2n+1}$ and reasoning as in (ii), $I^2 x_{2n+1} = S^2 x_{2n+1} = T^2 x_{2n+1}$.

Therefore it follows, there exists a point u such that $I^2 u = S^2 u = T^2 u$. If I weak** commutes with S, we have $d(S^2 Iu, IS^2 u, a) \leq d(SI^2, I^2 Su, a) \leq d(SIu, ISu, a) \leq d(S^2 u, I^2 u, a) = O$, which implies that

$S^2 Iu = IS^2 u$, $SI^2 u = I^2 Su$, IS_u and so $I^2 Su = S^3 u$ …………..(5)

Thus, $d(I^2 Su, S^2 Su, a) + d(I^2 u, T^2u, a) = O$ and using condition (2) we deduce that $Iu = IS^2 u = S^2 Iu = T^2 u = z$ and using (I) on the assumption $T^2 x \neq z$, we have $d(x, T^2 x, a) = d(S^2 x, T^2 x, a)$

$$\leq (a+\beta) \frac{d(I^2x, S^2x, a)\, d(I^2x, T^2x, a)}{d(I^2x, S^2x, a) + d(T^2x, T^2x, a)}$$

a contradiction, So $x = T^2 x$. Now using the rotativity of T with respect to I (or with respect to S) we have $d(T_x, x, a) = d(T_x, I^2 x,) \leq d(Ix, T^2 x, a) = d(x, x, a) = O$ and so z is a common fixed point of I,S and T. Similarly it can be proved if we assumed that I weak** commite with T and rotativity of S with respect to I (or with respect to T). Now suppose that z_1 in another common fixed point of I and S. Then

$d(I^2 x, S^2 x_1, a) + d(I^2 x, T^2 x, a) = O$ and condition (2) implies that $z_1 = Sz_1 = S^2 x_1 = T^2 z = z$. We can similarly prove that z is the unique common fixed point of I and T. //

(1.10) **THEOREM:** Let (X,d) be a complete 2 – metric space, d continuous and A.S.T be three mappings of X into X. Let S^2 (X) UT^2(X) $\subseteq$ A^2 (X) and the pairs of mappings {A,S} and {A,T} be weak** commutating and

$d(S^2 x, T^2 y,a) \le a \dfrac{[d(A^2x,S^2x,a)]^{r+w}[d(A^2 y,T^2 y,a)]^{1-r}}{[d(A^2x,A^2y,a)]^w} + \beta\, [d(A^2 x, A^2 y, a)]^{1-r-w} [d(S^2 x, T^2 y, a)]^{r+w}$ for all x,y, a in X with Ax ≠ Ay and for some a,β, γ ε (0,1), γ ε (O,1) with O < (a+β) < 1 and 2r + w = 1 when w ≠ O. Then A,S,T have a unique common fixed point if either of A,S, T is continuous.

Proof: For any X_o ε X , we have x_1 ε $x_,$ such that $S^2 x_o = A^2 x_1$ for $S^2(x) \subseteq A^2(X)$. Similarly for this x_1 we get x_2 ε X such that $T^2 x_1 = A^2 x_2$ (say) for $T^2(X) \subseteq A^2(X)$ and so on. Inductively we have a common sequence $\{y_n\}$ defined as $y_{2n+1} = A^2 x_{2n+1} = S^2 x_{2n}$ $y_{2n+2} = A^2 x_{2n+2} = T^2 x_{2n+1}$, for n = o,1,2 …………………

$d(y_{2n,\ 2n+1}, a) = d(A^2 x_{2n,} A^2 x_{2n+1}, a) = d(T^2 x_{2n-1}, S^2 x_{2n}, a) = d(S^2 x_{2n}, T^2 x_{2n-1}, a)$

$$\le a \frac{[d(A^2x_{2n},S^2x_{2n},a)]^{r+w}[d(A^2x_{2n},T^2x_{2n},T^2x_{2n-1},a)]^{1-r}}{[d(A^2x_{2n},A^2x_{2n-1},a)]^w}$$

$$+\beta[d(Ax_{2n},Ax_{2n-1},a)^{1-r-w}[d(A^2x_{2n},T^2x_{2n-1},a)]^{r+w}$$

Or, $d(y_{2n}, y_{2n+1}, a) \le (a+\beta)^{\frac{1}{1-r-w}}$ $(d(y_{2n,} y_{2n-1}, a)]$

Or, $d_{2n} \le (a+\beta)^{\frac{1}{1-r-w}} d_{2n-1}$

Where $d_{2n} = d(y_{2n}, y_{2n+1,} a)$

$d_{2n-1} = d(y_{2n}, y_{2n+1}, a)$

Again,

$d_{2n+1} = d(y_{2n+1}, y_{2n+2}, a) = d(A^2 x_{2n+1}, A^2 x_{2n+2,} a)$
$= d(S^2 x_{2n}, T^2 x_{2n+1}, a)$

$$\le a \frac{[d(A^2x_{2n},S^2x_{2n},a)]^{r+w}[d(A^2x_{2n},T^2x_{2n},T^2x_{2n-1},a)]^{1-r}}{[d(A^2x_{2n},A^2x_{2n-1},a)]^w}$$

$$+\beta[d(Ax_{2n},Ax_{2n-1},a)^{1-r-w}[d(A^2x_{2n},T^2x_{2n-1},a)]^{r+w}$$

Or, $d_{2n+1} = a[d_{2n}]^r [d_{2n+1}]^{1-r} + \beta [d_{2n}]^{1-r-w} [d_{2n+1}]^{r+W}$

<u>Case-I:</u> When w ≠ O then we obtain

$d_{2n} = (a+\beta)^{1/r} d_{2n-1}$

and $d_{2n+1} \le a[d_{2n}]^r [d_{2n+1}]^{1-r} + \beta [d_{2n}]^r [d_{2n+1}]^{1-r}$ or, $d_{2n+1} \le (a+\beta)^{1/r} [d_{2n}]$.

Case-II: when w = O, we get

$d_{2n} \le (a+\beta)^{1/1-r} d_{2n-1}$

and $d_{2n+1} \leq a [d_{2n}]^r [d_{2n+1}]^{1-r} + \beta [d_{2n}]^{1-r} [d_{2n+1}]^r$

we claim that $d_{2n+1} \leq d_{2n}$. If it is not so suppose that
$d_{2n+1} \geq d_{2n}$ then we have
$d_{2n+1} \leq a[d_{2n+1}]^r [d_{2n+1}]^{1-r} + \beta [d_{2n+1}]^{1-r} [d_{2n+1}]^r$

$= (a+\beta) d_{2n+1}$, which is a contradiction.

Since $(a + \beta) < 1$.

Therefore, $d_{2n+1} \leq d_{2n} \leq q d_{2n-1} \leq q^d{}_{2n-2} \leq q^2 d_{2n-3} \leq \ldots\ldots\ldots\ldots \leq q^n d_o \to O$ as $n \to \infty$.

Since $q = (a+\beta)^{\frac{1}{1-r}} < 1$. Thus in both cases the sequence $\{y_n\}$ is a Cauchy sequence. Therefore $\{A^2 x_{2n}\}$ is a Cauchy sequence and so coverges to a point u in X. Also,

$$\underset{n \to \infty}{Lt} \{A^2 x_{2n}\} = \{S^2 x_{2n-1}\} \text{ and } \{A^2 x_{2n+1}\} = \{T^2 x_{2n}\}$$

Converges to u because they are subsequence of the sequence $\{A^2 x_{2n}\}$.

First let A is continuous then the sequence $\{A^4 x_{2n}\}$ and $\{A^2 S^2 x_{2n}\}$ converge to a point A^2 u. A weak** commute with S we have

$$\begin{aligned} d(S^2 A^2 x_{2n}, A^2 u, a) \;&X\; d(S^2 A^2 x_{2n}, A^2 u, A^2S^2 x_{2n}) \\ &+ d(S^2 A^2 x_{2n}, A^2 S^2 x_{2n}, a) + d(A^2 S^2 x_{2n}, A^2 u, a) \\ &\leq d(S^2 x_{2n}, A^2 x_{2n}, A^2 u) + d(A^2 x_{2n}, S^2 x_{2n}, a) + d(A^2 S^2 x_{2n}, A^2 u, a) \end{aligned}$$

Which imply on letting $n \to \infty$ that $\{S^2 A^2 x_{2n}\}$ also converges to A^2 u. Now we clain that $T^2 u = A^2 u$. Suppose not then we have

$$d(S^2 A^2 x_{2n}, T^2 u, a) \leq a \frac{[d(A^4x_{2n}, S^2A^2x_{2n}, a)]^{r+w}[d(A^2u, T^2u, a)]^{1-r}}{[d(A^4x_{2n}, A^2u, a)]^w} + \beta[(A^4x_{2n}, A^2u, a)]^{1-r-w}[d(A^2u, T^2u, a)]^{1-r}$$

when $n \to \infty$, we deduce that $d(A^2 u, T^2 u, a) < O$, a contradiction. So $T^2 u = A^2 u$.

Now suppose that $S^2 u \neq T^2 u$, then

$$d(S^2 u, T^2 u, a) \leq a \frac{[d(A^2u, S^2u, a)]^{r+w} [d(A^2u, T^2u, a)]^{1-r}}{[d(A^2u, A^2u, a)]^w} + \beta[d(A^2u, A^2u, a)]^{1-r-w}[d(S^2u, T^2u, a)]^{r+w} = O, \textit{contradiction.} \text{ So } S^2 u = T^2 u.$$

So, we have $S^2 u = T^2 u = A^2 u$.

A similar conclusion we get if we assume A is weak** commute with T.

Now the sequence $\{S^2 Ax_{2n}\}$ converge to Au. A being weak**

commute with S, we deduce that

$d(S^2 Ax_{2n}, Au, a) \le d(S^2 Ax_{2n}, A S^2 x_{2n}, Su) + d(S^2 Ax_{2n}, AS^2 x_{2n}, a)$

$+ d(AS^2 x_{2n}, Su, a)$

$\le d(S^2 x_{2n}, A^2 x_{2n}, a) + d(A^2 x_{2n}, S^2 x_{2n}, a)$

$+ d(SA^2 x_{2n}, Su, a)$

Which implies as $n \to \infty$ that $\{A^2 Sx_{2n}\}$ converge to Su,

Now using this we have

$$d(S^2 Ax_{2n}, T^2 x_{2n+1}, a) \le a \frac{[d(A^3x_{2n,}S^2 Ax_{2n}, a)]^{r+w}[d(A^2x_{2n+1}T^2x_{2n+1}, T^2x_{2n+1}, a]^{1-r}}{[d(A^3x_{2n}, A^2x_{2n+1}, a)]^w} + \beta[d(A^3x2n, A^2x_{2n+1}, a)]^{1-r-w}[d(S^2Ax_{2n}, T^2x_{2n+1}, a)]^{r-w}$$

letting $n \to \infty$,

$d(Au, u, a) \le \beta\ d(Au, u, a)$,a contradiction,

Thus $Au = u$ so $A^2 u = u$.

Thus we conclude that Au = Su = Tu = u. Therefore u is the common fixed point of A,S and T.

Now suppose that S is continuous, then the sequence $\{A^2 Sx_{2n}\}$ converges to Su, A being weak** commutative we have.

$d(A^2 Sx_{2n}, Su, a) \le d(A^2 Sx_{2n,}Su, SA^2 x_{2n}) + d(AS^2 x_{2n}, SA^2 x_{2n},a) + d(SA^2 x_{2n}, Su, a)$
$\le d(S^2 x_{2n} + Su, A^2 x_{2n}) + d(A^2 u, S^2 u, a) + d(SA^2 x_{2n},Su, a)$

letting $n \to \infty$ we observe that $\{A^2 Sx_{2n}\}$ converges to Su. Now,

$$d(S^2 x_{2n},T^2 x_{2n+1}, a) \leq a \frac{[d(A^2Sx_{2n}, S^3x_{2n,}a)^{r+w}[d(A^2x_{2n+1}\, T^2\, x_{2n+1}\, T^2x_{2n+1},a)]^{1-r}}{[d(A^2Sx_{2n,}\, A^2x_{2n},a)]^w} + \beta[d(A^2Sx_{2n}, A^2\, x_{2n-1}, a)]^{1-r-w}[d(S^2x_{2n}, T^2x_{2n+1}, a)]^{r+w}$$

letting $n \to \infty$, d(Su, u, a) ≤ β d(Su, u,a), a contradiction, Thus Su = u and so S^2 u = u.

As above we can show that T^2 u = A^2 u = S^2 u. Thus again we have u = Tu = Su = Au, i.e, u is the common fixed point of A,S,T.

If the mapping T is continuous instead of S analogously we can show that u is the common fixed point of A,S,T.

Now we claim that u is the unique common fixed point of A,S and T. For this let v ≠ u be another common fixed point, then

$$D(u,v,a) = d(S^2\, u, T^2\, v, a) \leq a \frac{[d(A^2Sx_{2n,}S^3x_{2n},a)]^{r+w}[d(A^2x_{2n+1}\, T^2\, x_{2n+1}T^2x_{2n+1},a)^{1-r}]}{[d(A^2Sx_{2n}, A^2x_{2n},a)]^w} + \beta[d(A^2Sx_{2n}, A^2x_{2n-1},a)]^{1-r-w}[d(S^2x_{2n}, T^2x_{2n+1},a)]^{r+w}$$

Letting $n \to \infty$, d(Su, u,a) ≤ β d(Su, u, a), a contradiction, Thus Su = u and so S^2 u = u.

As above we can show that T^2 u = A^2 u = S^2 u. Thus again we have u = Tu = Su = Au, i.e. u is the common fixed point of A, S, T.

If the mapping T is continuous instead of S analogously we can show that u is the common fixed point of A,S, T.

Now we claim that u is the unique common fixed point of A,S and T. For this let v ≠ u be another common fixed point, then

$$d(u,v,a) = d(S^2\, u, T^2\, v, a) \leq a \frac{[d(A^2u, S^2u,a)]^{r+w}[d(A^2v, T^2v,a)]^{1-r}}{[d(A^2u, A^2v,a)]^w} + \beta\, [d(A^2\, u, A^2\, v, a)]^{1-r-w}\, [d(S^2\, u, T^2\, v, a)]^{r+w} = \beta\, [d(u,v, a)]^{1-r-w}\, [d(u,v,a)]^{r+w}$$

∴ d(u,v,a) ≤ β d (u,v,a), which is a contradiction. Thus u is the unique common fixed point of A,S and T. //

(1.11) <u>THEOREM:</u> Let A,S and T be threo self-mappings of a complete 2-metric space

(X,d) with d continuous. Let $S^2(X) \cup T^2(X) \subseteq A^2(X)$. If {A,S} and {A,T} be weak** commutating pairs and the following condition holds:
(I) $d(S^2 x, T^2 y, a) \leq a\,[d(A^2 x, S^2 x, a)]^{1-\beta-r}\,[d(A^2 y, T^2 y, a)]^{\beta}\,[d(A^2 x, A^2 y, a)]^{r}$ for all x,y,a in X, $O \leq a \leq 1$, $O \leq \beta \leq 1$, $O \leq \gamma$ with $O < \beta + \gamma \leq 1$. If may of A,S and T are continuous then A,S and T have a unique common fixed point.

Proof: For any arbitrary point $x_o \varepsilon X$. Construct the sequence $\{y_n\}$ as in Theorem (1.10). Using (I) it is easy to show that sequence $\{y_n\}$ is a Cauchy sequence. Since X is complete, so there exists a point u in X. Now, let A is continous, then the sequence $\{A^4 x_{2n}\}$ and $\{A^2 S^2 x_{2n}\}$ converge to a point $A^2 u$. A is weak** commutative with S, we have $d(S^2 A^2 x_{2n}, A^2 u, a) \leq d(S^2 A^2 x_{2n}, A^2 u, A^2 S^2 x_{2n}) + d(S^2 A^2 x_{2n}, A^2 S^2 x_{2n}, a) + d(A^2 S^2 x_{2n}, A^2 u, a) \leq d(S^2 x_{2n}, A^2 x_{2n}, A^2 u) + d(A^2 x_{2n}, S^2 x_{2n}, a) + d(A^2 S^2 x_{2n}, A u, a)$

which imply on letting $n \to \infty$ that $\{S^2 A^2 x_{2n}\}$ also converges to $A^2 u$. Now we claim that $T^2 u = A^2 u$. Suppose not then we have

$d(S^2 A^2 x_{2n}, T^2 u, a) \leq a\,[d(A^4 x_{2n}, S^2 A^2 x_{2n}, a)]^{1-\beta-y}{}_x\,[d(A^2 u, T^2 u, a)]^{\beta}\,[d(A^2 S^2 x_{2n}, A^2 u, a)]^{\gamma}$

when $n \to \infty$, we deduce that $d(A^2 u, T^2 u, a) \leq O$, a contradiction. So $T^2 u = A^2 u$. Now suppose that $S^2 u \neq T^2 u$. then

$d(S^2 u, T^2 u, a) \leq a\,[d(A^2 u, S^2 u, a)]^{1-\beta-y}\,[d(A^2 u, T^2 u, a)]^{\beta}\,[d(A^2 u, A^2 u, a)]^{\gamma}$
$= O$, a contradiction. So, $S^2 u = T^2 u$.

Thus we have $S^2 u = T^2 u = A^2 u$.

A similar conclusion we get if we assume that A is weak** commute with T. A being weak** commutative with S, we deduce that

$d(S^2 Ax_{2n}, Au, a) \leq d(S^2 Ax_{2n}, AS^2 x_{2n}, Su)$
$+ d(S^2 Ax_{2n}, AS^2 x_{2n}, a) + d(AS^2 x_{2n}, Su, a)$
$\leq d(S^2 x_{2n}, A^2 x_{2n}, Su) + d(S^2 x_{2n}, A^2 x_{2n}, a)$
$+ d(SA^2 x_{2n}, Su, a)$

Which implies that us $n \to \infty$ that $\{A^2 Sx_{2n}\}$ converges to Su. Now being this we have
$d(S^2 Ax_{2n}, T^2 x_{2n+1}, a) \leq a\,[d(A^3 x_{2n}, S^2 Ax_{2n}, a)]^{1-\beta-\gamma}$
$[d(A^2 x_{2n+1}, T^2 x_{2n+1}, a)]^{\beta}\ x\ [d(A^3 x_{2n}, A^2 x_{2n+1}, a)]^{\gamma}$.

Letting $n \to \infty$, $d(Au, u, a) \leq \beta\ d(Au, u, a)$, a

Contradiction. Thus Au = u and so A^2 u = u. Similarly as done above Su = Tu = u.

Therefore u is the common fixed point of A,S and T.

Now suppose that S is continuous then the sequence $\{A^2 Sx_{2n}\}$ converges to Su. A being weak** commute we have

$d(A^2 Sx_{2n}, Su, a) \le d(A^2 Sx_{2n}, Su, SA^2 x_{2n}) + d(A^2 Sx_{2n}, SA^2 x_{2n}, a) + d(SA^2 x_{2n}, Su, a)$
$\le d(S^2 x_{2n}, Su, A^2 x_{2n}) + d(A^2 u, S^2 u, a) + d(SA^2 x_{2n}, Su, a)$

Letting $n \to \infty$ we observe that $\{A^2 Sx_{2n}\}$ converges to Su.
Now,

$D(S^3 x_{2n}, T^2 x_{2n+1}, a) \le a ([d(A^2 Sx_{2n}, S^3 x_{2n}, a)]^{1-\beta-y} [d(A^2 x_{2n+1}, T^2 x_{2n+1}, a)]^{\beta}$ x $[d(A^2 Sx_{2n}, A^2 x_{2n+1}, a)]^{\gamma}$,

Letting $n \to \infty$, $d(Su, u, a) \le \beta\ d(Su, u, a)$, a contradiction. Thus Su = u and S^2 u = u.
Suppose S^2 u $\neq$ T^2 u. Then

$D(S^2 u, T^2 u, a) \le a [d(A^2 u, S^2 u, a)]^{1-\beta-\gamma} [d(A^2 u, T^2 u, a)]^{\beta}$ x $[d(A^2 u, A^2 u, a)]^{\gamma}$
= O, a contradiction. So S^2 u = T^2 u.

Since A is weak** commutative with S, we have S^2 Au = AS^2 u and S^2 Au = A^3 u = Au.
Now, $d(Au, u, a) = d(A^3 u, T^2 u, a)$
$\le a [d(A^3 u, S^2 Au, a)]^{1-\beta-\gamma} [d(A^2 u, T^2 u, a)]^{\beta} [d(A^3 u, Au, a)]^{\gamma}$

= O, which implies that Au = u.
Therefore Au = Su = u.

Since A is weak** commutative with T, we have A^2 Tu = TA^2 u and T^2 Au = T^3 u.
Now,

$D(Tu, u,a) = d(T^3 u, S^2 u, a) = d(S^2 u, T^2 Tu, a)$
$\le a [d(A^2 u, S^2 u, a)]^{1-\beta-\gamma} [d(A^2 Tu, T^3 u, a)]^{\beta}$ x $[d(A^2 u, A^2 Tu, a)]^{\gamma}$

= O, which implies that Tu = u.
Therefore Au = Su = Tu = u i.e., u is the common fixed point of A, S and T.

Analogously we can prove if we assume T is continuous instead of S.
Now we claim that u is the unique common fixed point of A,S and T. For this let v $\neq$ u be another common fixed point. Then

$D(u, v,a) = d(S^2 u, T^2 v, a)$

$\leq a [d(A^2 u, S^2 u, a)]^{1-\beta-\gamma} [d(A^2 (v), T^2 (v), a)]^{\beta} x [d(A^2(u), A^2(v), a)]^{\gamma}$
i.e., $d(u, v,a) \leq O$, which implies that $u = v$. Thus u is the unique common fixed point of

A,S and T. //

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