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Primary Decomposition in A Γ-Semigroup

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ABSTRACT

In this paper the terms P-primary, primary decomposition of a **Γ**-ideal, reduced primary decomposition of a **Γ**-ideal in a **Γ**-semigroup S are introduced. If A_1, A_2, …A_n are P-primary **Γ**-ideals in a **Γ**-semigroup S, then it is proved that $\bigcap_{i=1}^{n} A_i$ is also a P-primary **Γ**-ideal. If a **Γ**-ideal A in a **Γ**-semigroup S has a primary decomposition, then it is proved that A has a reduced primary decomposition. Further it is proved that every Γ-ideal in a (left, right) duo noetherian Γ-semigroup S has a reduced (right, left) primary decomposition. If A and B are two Γ-ideals in a Γ-semigroup S, then it is proved that $A^l(B) = \{ x \in S : < x >\Gamma B \subseteq A\}$ and $A^r(B) = \{ x \in S : B\,\Gamma < x > \subseteq A\}$ are Γ-ideals of S containing A. Further it is proved that (1) if A is a left primary **Γ**-ideal of a **Γ**-semigroup S, then $A^l(B)$ is a left primary **Γ**-ideal, (2) if A is a right primary **Γ**-ideal of a **Γ**-semigroup S, then $A^r(B)$ is a right primary **Γ**-ideal. It is proved that if Q is a P-primary **Γ**-ideal and if $A \not\subseteq P$, then $Q^l(A) = Q^r(A) = Q$ and also if $A \subseteq P$ and $A \not\subseteq Q$, then $\sqrt{(Q^l(A))} = \sqrt{(Q^r(A))} = \sqrt{Q}$. If $A_1, A_2, \ldots, A_n$, B are **Γ**-ideals of a **Γ**-semigroup S, then it is proved that $\left(\bigcap_{i=1}^{n} A_i\right)^l (B) = \bigcap_{i=1}^{n} (A_i)^l (B)$. Further if a **Γ**-ideal A in a **Γ**-semigroup S has two reduced (one sided) primary decompositions; $A = A_1 \cap A_2 \cap \ldots \cap A_k = B_1 \cap B_2 \cap \ldots B_s$, where A_i is P_i -primary and B_j is Q_j-primary, then it is proved that k = s and after reindexing if necessary $P_i = Q_i$ for $i = 1, 2, \ldots, k$.

SUBJECT CLASSIFICATION (2010) : 20M07, 20M11, 20M12.

KEY WORDS : P- primary Γ-ideal, left primary decomposition, right primary decomposition, primary decomposition, reduced primary decomposition.

1. INTRODUCTION :

Γ-semigroup was introduced by Sen and Saha [16] as a generalization of semigroup. Satyanarayana[14], [15] initiated the study of primary ideals in commutative semigroups and obtained primary decomposition theorem in commutative noetherian semigroups. Anjaneyulu. A [1], [2] and [3] initiated the study of primary ideals, semiprimary ideals in general semigroups and obtained primary decomposition theorem for ideals in a duo noetherian semigroups. Madhusudhana Rao, Anjaneyulu and Gangadhara Rao [8], and [11] initiated the study of Γ-ideals, prime Γ-radicals, Primary Γ-ideals and semiprimary Γ-ideals in Γ-semigroups. In this paper we establish a ' primary decomposition theorem` in duo noetherian Γ-semigroups. Also we obtain a necessary condition to have a unique reduced primary decomposition for a Γ-ideal in an arbitrary Γ-semigroup.

2. PRELIMINARIES :

DEFINITION 2.1 : Let S and Γ be any two non-empty sets. Then S is said to be a **Γ-*semigroup*** if there exist a mapping from $S \times \Gamma \times S$ to S which maps $(a, \gamma, b) \rightarrow a\gamma b$ satisfying the condition : $(a\alpha b)\beta c = a\alpha(b\beta c)$ for all $a, b, c \in S$ and $\alpha, \beta \in \Gamma$.

NOTE 2.2 : Let S be a Γ-semigroup. If A and B are two subsets of S, we shall denote the set $\{ a\gamma b : a \in A , b \in B$ and $\gamma \in \Gamma \}$ by AΓB.

DEFINITION 2.3 : A Γ-semigroup S is said to be ***commutative*** **Γ-semigroup** provided $a\gamma b = b\gamma a$ for all $a,b \in S$ and $\gamma \in \Gamma$.

NOTE 2.4 : If S is a commutative Γ-semigroup then $a\Gamma b = b\Gamma a$ for all $a, b \in S$.

NOTE 2.5 : Let S be a Γ-semigroup and $a, b \in S$ and $\alpha \in \Gamma$. Then $a\alpha a\alpha b$ is denoted by $(a\alpha)^2 b$ and consequently $a\, \alpha\, a\, \alpha\, a\, \alpha \ldots (n\ terms)b$ is denoted by $(a\alpha)^n b$.

DEFINITION 2.6 : A nonempty subset A of a Γ-semigroup S is said to be a ***left* Γ-*ideal*** of S if $s \in S, a \in A, \alpha \in \Gamma$ implies $s\alpha a \in A$.

NOTE 2.7 : A nonempty subset A of a Γ-semigroup S is a left Γ- ideal of S iff $S\Gamma A \subseteq A$.

DEFINITION 2.8 : A nonempty subset A of a Γ-semigroup S is said to be a ***right* Γ-*ideal*** of S if $s \in S, a \in A, \alpha \in \Gamma$ implies $a\alpha s \in A$.

NOTE 2.9 : A nonempty subset A of a Γ-semigroup S is a right Γ- ideal of S iff $A\Gamma S \subseteq A$.

DEFINITION 2.10 : A nonempty subset A of a Γ-semigroup S is said to be a ***two sided* Γ- *ideal*** or simply a **Γ- *ideal*** of S if $s \in S$, $a \in A$, $\alpha \in \Gamma$ imply $s\alpha a \in A$, $a\alpha s \in A$.

NOTE 2.11 : A nonempty subset A of a Γ-semigroup S is a two sided Γ-ideal iff it is both a left Γ-ideal and a right Γ- ideal of S.

THEOREM 2.12 : The nonempty intersection of any two (left or right) Γ-ideals of a Γ-semigroup S is a (left or right) Γ-ideal of S**.**

THEOREM 2.13 : The nonempty intersection of any family of (left or right) **Γ**-ideals of a **Γ**-semigroup S is a (left or right) **Γ**-ideal of S**.**

THEOREM 2.14 : The union of any two (left or right) **Γ**-ideals of a **Γ**-semigroup S is a (left or right) **Γ**-ideal of S.

THEOREM 2.15 : The union of any family of (left or right) **Γ**-ideals of a **Γ**-semigroup S is a (left or right) **Γ**-ideal of S.

DEFINITION 2.16 : A Γ-ideal A of a Γ-semigroup S is said to be a ***proper* Γ*-ideal*** of S if A is different from S.

DEFINITION 2.17 : A Γ-ideal A of a Γ-semigroup S is said to be a ***trivial* Γ*-ideal*** provided S\A is singleton.

DEFINITION 2.18 : A Γ-ideal A of a Γ-semigroup S is said to be a ***maximal* Γ*-ideal*** provided A is a proper Γ-ideal of S and is not properly contained in any proper Γ-ideal of S.

THEOREM 2.19 : If S is a **Γ**-semigroup with unity 1 then the union of all proper **Γ**-ideals of S is the unique maximal **Γ**-ideal of S.

DEFINITION 2.20 : A Γ- semigroup S is said to be a ***left duo* Γ*- semigroup*** provided every left Γ- ideal of S is a two sided Γ- ideal of S.

DEFINITION 2.21 : A Γ- semigroup S is said to be a ***right duo* Γ*- semigroup*** provided every right Γ-ideal of S is a two sided Γ- ideal of S.

DEFINITION 2.22 : A Γ- semigroup S is said to be a ***duo* Γ*- semigroup*** provided it is both a left duo Γ- semigroup and a right duo Γ- semigroup.

THEOREM 2.23 : A Γ-semigroup S is a duo Γ-semigroup if and only if $x\Gamma S^1 = S^1\Gamma x$ for all $x \in S$.

DEFINITION 2.24 : A Γ- ideal P of a Γ-semigroup S is said to be a ***completely prime* Γ*- ideal*** provided $x, y \in S$ and $x\Gamma y \subseteq P$ implies either $x \in P$ or $y \in P$.

DEFINITION 2.25 : A Γ- ideal P of a Γ-semigroup S is said to be a ***prime* Γ*- ideal*** provided A, B are two Γ-ideals of S and $A\Gamma B \subseteq P \Rightarrow$ either $A \subseteq P$ or $B \subseteq P$.

THEOREM 2.26 : A Γ- ideal P of a Γ-semigroup S is a prime Γ- ideal iff $a, b \in S$ such that $a\Gamma S^1\Gamma b \subseteq P$, then either $a \in P$ or $b \in P$.

THEOREM 2.27 : Every completely prime **Γ**-ideal of a **Γ**-semigroup S is a prime **Γ**-ideal of S.

THEOREM 2.28 : Let S be a commutative Γ-semigroup. A Γ-ideal P of S is prime Γ-ideal if and only if P is a completely prime Γ-ideal.

DEFINITION 2.29 : A Γ-ideal A of a Γ-semigroup S is said to be a ***completely semiprime* Γ- *ideal*** provided $x\Gamma x \subseteq A$; $x \in S$ implies $x \in A$.

THEOREM 2.30 : Every completely prime Γ-ideal of a Γ-semigroup S is a completely semiprime Γ-ideal of S.

THEOREM 2.31 : The nonempty intersection of any family completely prime Γ-ideals of a Γ-semigroup S is a completely semiprime Γ-ideal of S.

DEFINITION 2.32 : A Γ- ideal A of a Γ-semigroup S is said to be a ***semiprime* Γ- *ideal*** provided $x \in S$, $x\Gamma S^1\Gamma x \subseteq A$ implies $x \in A$.

THEOREM 2.33 : Every completely semiprime Γ-ideal of a Γ-semigroup S is a semiprime Γ-ideal of S.

THEOREM 2.34 : Let S be a commutative Γ-semigroup. A Γ-ideal A of S is completely semiprime iff semiprime.

THEOREM 2.35 : Every prime Γ-ideal of a Γ-semigroup S is a semiprime Γ-ideal of S.

THEOREM 2.36 : The nonempty intersection of any family of prime Γ-ideals of a Γ-semigroup S is a semiprime Γ-ideal of S.

NOTATION 2.37 : If A is a Γ-ideal of a Γ-semigroup S, then we associate the following four types of sets.

A_1 = The intersection of all completely prime Γ-ideals of S containing A.

$A_2 = \{x \in S : (x\Gamma)^{n-1}x \subseteq A$ for some natural number n }

A_3 = The intersection of all prime ideals of S containing A.

$A_4 = \{x \in S : (< x >\Gamma)^{n-1} < x > \subseteq A$ for some natural number n }

THEOREM 2.38 : If A is a Γ- ideal of a Γ-semigroup S, then $A \subseteq A_4 \subseteq A_3 \subseteq A_2 \subseteq A_1$.

THEOREM 2.39 : If A is a Γ-ideal in a duo Γ-semigroup S then $A_1 = A_2 = A_3 = A_4$.

DEFINITION 2.40 : If A is a Γ-ideal of a Γ-semigroup S, then the intersection of all prime Γ-ideals of S containing A is called ***prime* Γ-*radical*** or simply **Γ-*radical*** of A and it is denoted by **√A** or ***rad A***.

DEFINITION 2.41 : If A is a Γ-ideal of a Γ-semigroup S, then the intersection of all completely prime Γ-ideals of S containing A is called ***complete prime* Γ-*radical*** or simply ***complete* Γ-*radical*** of A and it is denoted by ***c. rad A***.

NOTE 2.42 : If A is a Γ-ideal of a Γ-semigroup S then ***rad* A = A_3** and ***c.rad* A = A_4.**

THEOREM 2.43 : If A is a Γ-ideal of a duo Γ-semigroup S, then *rad* A = *c.rad* A

THEOREM 2.44 : If A and B are any two Γ-ideals of a Γ-semigroup S, then

(i) A ⊆ B ⇒ √(A) ⊆ √(B) .
(ii) √(AΓB) = √(A∩B) = √(A) ∩ √(B).
(iii) √(√(A)) = √(A).

THEOREM 2.45 : If A and B are any two Γ-ideals of a Γ-semigroup S, then

(i) A ⊆ B ⇒ *c.rad* A ⊆ *c.rad* B.
(ii) *c.rad* (AΓB) = *c.rad* (A∩B) = *c.rad* (A) ∩ *c.rad* (B).
(iii) *c.rad* (*c.rad* A) = *c.rad* A.

3. <u>PRIMARY IDEALS:</u>

DEFINITION 3.1 : A Γ-ideal A of a Γ-semigroup S is said to be a ***left primary* Γ*-ideal*** provided

i) If X, Y are two Γ-ideals of S such that XΓY ⊆ A and Y ⊈ A then X ⊆ √A.
ii) √A is a prime Γ-ideal of S.

DEFINITION 3.2: A Γ-ideal A of a Γ-semigroup S is said to be a ***right primary* Γ*-ideal*** provided

i) If X, Y are two Γ-ideals of S such that XΓY ⊆ A and X ⊈ A then Y ⊆ √A.
ii) √A is a prime Γ-ideal of S.

DEFINITION 3.3 : A Γ-ideal A of a Γ-semigroup S is said to be a ***primary* Γ*-ideal*** provided A is both a left primary Γ-ideal and a right primary Γ-ideal.

THEOREM 3.4 : Let A be a Γ-ideal of a Γ-semigroup S. Then X, Y are two Γ-ideals of S such that XΓY ⊆ A and Y ⊈ A ⇒ X ⊆ √A if and only if *x*, *y* ∈ S, < *x* > Γ < *y* > ⊆ A and *y* ∉ A ⇒ *x* ∈ √A.

***Proof* :** Suppose that X, Y are two Γ-ideals of S such that XΓY ⊆ A, Y⊈A ⇒ X⊆√A.
Let *x*, *y* ∈ S, < *x* >Γ < *y* > ⊆ A and *y* ∉ A. Now *y* ∉ A ⇒ < *y* > ⊈ A.

By supposition < *x* > Γ < *y* > ⊆ A and < *y* > ⊈ A ⇒ < *x* > ⊆ √A. Therefore *x* ∈ √A.

Conversely suppose that *x*, *y* ∈ S, < *x* > Γ < *y* > ⊆ A and *y* ∉ A ⇒ *x* ∈ √A.

Let X, Y be two Γ-ideals of S such that XΓY ⊆ A and Y ⊈ A.

Suppose if possible X ⊈ √A. Then there exists *x* ∈ X such that *x* ∉√A.

Since Y ⊈ A, let *y* ∈Y so that *y* ∉ A.

Now < *x* > Γ < *y* > ⊆ XΓY ⊆ A and *y* ∉ A ⇒ *x* ∈ √A. It is a contradiction. Therefore X ⊆ √A.

THEOREM 3.5 : Let A be a Γ-ideal of a Γ-semigroup S. Then X, Y are two Γ-ideals of S such that XΓY ⊆ A and X ⊈ A ⇒ Y ⊆ √A if and only if *x*, *y* ∈ S, < *x* > Γ < *y* > ⊆ A and *x* ∉ A ⇒ *y* ∈ √A.

***Proof* :** Suppose that X, Y are two Γ-ideals of S such that XΓY⊆A, X⊈A ⇒ Y⊆ √A.
Let *x*, *y* ∈ S, < *x* >Γ < *y* > ⊆ A and *x* ∉ A. Now *x* ∉ A ⇒ < *x* > ⊈ A.

By supposition < *x* > Γ < *y* > ⊆ A and < *x* > ⊈ A ⇒ < *y* > ⊆ √A. Therefore *y* ∈ √A.
Conversely suppose that *x*, *y* ∈ S, < *x* > Γ < *y* > ⊆ A and *x* ∉ A ⇒ *y* ∈ √A.

Let X, Y be two Γ-ideals of S such that XΓY ⊆ A and X ⊈ A.
Suppose if possible Y ⊈ √A. Then there exists *y* ∈ Y such that *y* ∉√A.

Since X ⊈ A, let *x* ∈ X so that *x* ∉ A.
Now < *x* > Γ < *y* > ⊆ XΓY ⊆ A and *x* ∉ A ⇒ *y* ∈ √A. It is a contradiction. Therefore Y ⊆ √A.

THEOREM 3.6 : Let S be a commutative Γ-semigroup and A be a Γ-ideal of S. Then the following conditions are equivalent.

1) A is a primary Γ-ideal.

2) X, Y are two Γ-ideals of S, XΓY ⊆ A and Y ⊈ A ⇒ X ⊆ √A.

3) *x*, *y* ∈ S, *x*Γ*y* ⊆ A, *y* ∉ A ⇒ *x* ∈ √A.

***Proof*:** (1) ⇒ (2) : Suppose that A is a primary Γ-ideal of S.
Then A is a left primary Γ-ideal of S.

So by definition 3.1, we get X, Y are two Γ-ideals of S, XΓY ⊆ A, Y ⊈ A ⇒ X ⊆ √A.

(2) ⇒ (3): Suppose that X, Y are two **Γ**-ideals of S, X**Γ**Y ⊆ A and Y ⊈ A ⇒ X ⊆ √A.

Let *x*, *y* ∈ S, *x***Γ***y* ⊆ A and *y* ∉ A.

*x***Γ***y* ⊆ A ⇒ < *x* > Γ < *y* > ⊆ A. Also *y* ∉ A ⇒ < *y* > ⊈ A.

Now < *x* > Γ < *y* > ⊆ A and < *y* > ⊈ A. Therefore by assumption < *x* > ⊆ √A ⇒ *x* ∈ √A.

(3) ⇒ (1) : Suppose assume that *x*, *y* ∈ S, *x***Γ***y* ⊆ A, *y* ∉ A then *x* ∈ √A.

Let X, Y be two **Γ**-ideals of S, X**Γ**Y ⊆ A and Y ⊈ A.

Y ⊈ A ⇒ there exists *y* ∈ Y such that *y* ∉ A. Suppose if possible X ⊈ √A.

Then there exists *x* ∈X such that *x* ∉√A. Now *x***Γ***y* ⊆ X**Γ**Y ⊆ A.

Therefore *x***Γ***y* ⊆ A and *y* ∉ A, *x* ∉√A. It is a contradiction. Therefore X ⊆ √A.

Let *x*, *y* ∈ S, *x***Γ***y* ⊆ √A. Suppose that *y* ∉ √A.

Now *x***Γ***y* ⊆ √A ⇒ $(x\Gamma y\Gamma)^{m-1}(x\Gamma y)$ ⊆ A ⇒ $(x\Gamma)^{m-1}x\Gamma(y\Gamma)^{m-1}y$ ⊆ A.

Since *y* ∉ √A, $(y\Gamma)^{m-1}y$ ⊈ A.

Now $(x\Gamma)^{m-1}x\Gamma(y\Gamma)^{m-1}y$ ⊆ A, $(y\Gamma)^{m-1}y$ ⊈ A ⇒ $(x\Gamma)^{m-1}x$ ⊆ √A ⇒ *x* ∈ √(√A) = √A.

√A is a completely prime Γ-ideal and hence √A is a prime Γ-ideal.

Therefore A is a left primary Γ-ideal. Similarly A is a right primary Γ-ideal.

Hence A is a primary Γ-ideal.

NOTE 3.7 : In an arbitrary Γ-semigroup a left primary Γ-ideal is not necessarily a right primary Γ-ideal.

EXAMPLE 3.8 : Let S = {*a*, *b*, *c*} and Γ = {*x*, *y*, *z*}. Define a binary operation **.** in S as shown in the following table.

.	*a*	*b*	*c*
a	*a*	*a*	*a*
b	*a*	*a*	*a*
c	*a*	*b*	*c*

Define a mapping from S × Γ × S ⟶ S by $a\alpha b = ab$, for all $a, b \in$ S and $\alpha \in \Gamma$.

It is easy to see that S is a Γ-semigroup. Now consider the Γ-ideal, $< a > = S^1\Gamma a\Gamma S^1 = \{a\}$.

Let $p\Gamma q \subseteq < a >$, $p \notin < a > \Rightarrow q \in \sqrt{< a >} \Rightarrow (q\Gamma)^{n-1}q \subseteq < a >$ for some $n \in$ N.

Since $b\Gamma c \subseteq < a >$, $c \notin < a > \Rightarrow b \in \sqrt{< a >}$. Therefore $< a >$ is left primary.

If $b \notin < a >$ then $(c\Gamma)^{n-1}c \nsubseteq < a >$ for any $n \in$ N $\Rightarrow c \notin \sqrt{< a >}$.

Therefore $< a >$ is not right primary.

THEOREM 3.9 : Every Γ-ideal A in a Γ-semigroup S is left primary if and only if every Γ-ideal A satisfies condition (i) of definition 3.1.

Proof **:** If every Γ-ideal A of S is left primary, then clearly every Γ-ideal satisfies condition (i) of definition 3.1.

Conversely suppose that every Γ-ideal of S satisfies condition (i) of definition 3.1.
Let A be any Γ-ideal of S. Suppose that $x, y \in$ S and $< x > \Gamma < y > \subseteq \sqrt{A}$.

If $y \notin \sqrt{A}$, then by our supposition $x \in \sqrt{(\sqrt{A})} = \sqrt{A}$.

Therefore $\sqrt{A}$ is a prime Γ-ideal. Hence A is left primary.

THEOREM 3.10 : Every Γ-ideal A in a Γ-semigroup S is right primary if and only if every Γ-ideal A satisfies condition (i) of definition 3.2.

Proof **:** If every Γ-ideal A of S is right primary, then clearly every Γ-ideal satisfies condition (i) of definition 3.2.
Conversely suppose that every Γ-ideal of S satisfies condition (i) of definition 3.2.

Let A be any Γ-ideal of S. Suppose that $x, y \in$ S and $< x > \Gamma < y > \subseteq \sqrt{A}$.

If $x \notin \sqrt{A}$ then by our supposition $y \in \sqrt{(\sqrt{A})} = \sqrt{A}$.

Therefore $\sqrt{A}$ is a prime Γ-ideal. Hence A is left primary.

DEFINITION 3.11 : A Γ-semigroup S is said to be ***left primary*** provided every Γ-ideal of S is a left primary Γ-ideal of S.

DEFINITION 3.12 : A Γ-semigroup S is said to be ***right primary*** provided every Γ-ideal of S is a right primary Γ-ideal of S.

DEFINITION 3.13 : A Γ-semigroup S is said to be ***primary*** provided every Γ-ideal of S is a primary Γ-ideal of S.

THEOREM 3.14 : Let S be a Γ-semigroup with identity and let M be the unique maximal Γ-ideal of S. If $\sqrt{A} = M$ for some Γ-ideal of S, then A is a primary Γ-ideal.

Proof **:** suppose that $x, y \in S$, $< x > \Gamma < y > \subseteq A$ and $y \notin A$.

If $x \notin \sqrt{A}$ then $< x > \not\subseteq \sqrt{A} = M$.

By theorem 2.19, M is the union of all proper Γ-ideals of S, we have $< x > = S$ and hence $< y > = < x > \Gamma < y > \subseteq A$. It is a contradiction. Therefore $x \in \sqrt{A}$.

Let $x, y \in S$, $< x > \Gamma < y > \subseteq \sqrt{A}$ and $< y > \not\subseteq \sqrt{A}$.

Since M is the unique maximal Γ-ideal, we have $< x > = S$.

Hence $< y > = < x > \Gamma < y > \subseteq \sqrt{A}$. It is a contradiction. Therefore $< x > \subseteq \sqrt{A}$.

Similarly if $< x > \not\subseteq \sqrt{A}$, then $< y > \subseteq \sqrt{A}$ and hence $\sqrt{A} = M$ is a prime Γ-ideal.

Thus A is left primary. By symmetry it follows that A is right primary.

Therefore A is a primary Γ-ideal.

NOTE 3.15: If a Γ-semigroup S has no identity, then the theorem 3.14, is not true, even if the Γ-semigroup S has a unique maximal Γ-ideal. In example 3.8, $\sqrt{< a >} = M$ where $M = \{a, b\}$ is the unique maximal Γ-ideal. But $< a >$ is not a primary Γ-ideal.

THEOREM 3.16: If S is a Γ-semigroup with identity, then for any natural number *n*, $(M\Gamma)^{n-1}M$ is primary Γ-ideal of S where M is the unique maximal Γ-ideal of S.

Proof **:** Since M is the only prime Γ-ideal containing $(M\Gamma)^{n-1}M$, we have $\sqrt{((M\Gamma)^{n-1}M)} = M$ and hence by theorem 3.14, $(M\Gamma)^{n-1}M$ is a primary Γ-ideal.

NOTE 3.17: If S has no identity then theorem 3.16, is not true. In example 3.8, $M = \{a, b\}$ is the unique maximal Γ-ideal, but $M\Gamma M = \{a\}$ is not primary.

DEFINITION 3.18 : A Γ-ideal A of a Γ-semigroup S is said to be ***semiprimary*** provided $\sqrt{A}$ is a prime Γ-ideal of S.

DEFINITION 3.19 : A Γ-semigroup S is said to be a ***semiprimary*** **Γ-*semigroup*** provided every Γ-ideal of S is a semiprimary Γ-ideal.

THEOREM 3.20 : (1) Every left primary Γ-ideal of a Γ-semigroup is a semiprimary Γ-ideal (2) Every right primary Γ-ideal of a Γ-semigroup is a semiprimary Γ-ideal.

Proof **:** By the definition of a left primary **Γ**-ideal of a **Γ**-semigroup, every left primary **Γ**-ideal is a semiprimary **Γ**-ideal. By the definition of a right primary **Γ**-ideal of a **Γ**-semigroup, every right primary **Γ**-ideal is a semiprimary **Γ**-ideal.

4. PRIMARY DECOMPOSITION IN A Γ-SEMIGROUP :

DEFINITION 4.1 : Let P be any prime Γ-ideal in a Γ-semigroup S. A primary Γ-ideal A in S is said to be ***P-primary*** or P is a ***prime Γ-ideal belonging to A*** provided $\sqrt{A}$ = P.

THEOREM 4.2 : If $A_1, A_2, \ldots, A_n$ are P-primary Γ-ideals in a Γ-semigroup S, then $\bigcap_{i=1}^{n} A_i$ is also a P-primary Γ-ideal.

Proof **:** Let A = $\bigcap_{i=1}^{n} A_i$. Now $\sqrt{A} = \sqrt{\cap A_i} = \cap\sqrt{A_i}$ = P. So $\sqrt{A}$ is a prime Γ-ideal. Suppose $< a >\Gamma< b > \subseteq A$ and $b \notin A$. So $b \notin A_i$ for some i. Now Suppose $< a >\Gamma< b > \subseteq A_i$ and $b \notin A_i$. Since A_i is a P-primary Γ-ideal, we have $a \in \sqrt{A_i}$ = P = $\sqrt{A}$. So A is a left primary Γ-ideal. Similarly we can show that A is a right primary Γ-ideal. Thus A is a P-primary Γ-ideal.

DEFINITION 4.3 : A Γ-ideal A in a Γ-semigroup S is said to have a (***left***, ***right***) ***primary decomposition*** if $A = A_1 \cap A_2 \cap \ldots \cap A_n$ where each A_i is a (left, right) primary Γ-ideal. If no A_i contains $A_1 \cap A_2 \cap \ldots \cap A_{i-1} \cap A_{i+1} \cap \ldots \cap A_n$ and the Γ-radicals P_i of the Γ-ideals A_i are all distinct, then the primary decomposition is said to be ***reduced***. If P_i is minimal in the set $\{ P_1, P_2, \ldots, P_n \}$ then P_i is said to be ***isolated prime***.

THEOREM 4.4 : If a Γ-ideal A in a Γ-semigroup S has a primary decomposition, then A has a reduced primary decomposition.

Proof **:** If $A = A_1 \cap A_2 \cap \ldots \cap A_n$ where each A_i is primary and some A_i contains $A_1 \cap A_2 \cap \ldots \cap A_{i-1} \cap A_{i+1} \cap \ldots \cap A_n$, then $A = A_1 \cap A_2 \cap \ldots \cap A_{i-1} \cap A_{i+1} \cap \ldots \cap A_n$ is also a primary decomposition. By thus eliminating the superfluous A_i reindexing we have $A = A_1 \cap A_2 \cap \ldots \cap A_k$ with no A_i containing the intersection of other A_j. Let $P_1, P_2, \ldots, P_r$ be the distinct prime Γ-ideals in the set $\sqrt{A_1}, \sqrt{A_2}, \ldots, \sqrt{A_k}$. Let A_i^1, $1 \leq I \leq r$ be the intersection of all A_j\`s belonging to the prime P_i. By theorem 4.2, each A_i^1 is primary for P_i. Clearly no A_i^1contains the intersection of all other A_j^1. Therefore $A = \bigcap_{i=1}^{n} A_i = \bigcap_{i=1}^{r} A_i^1$ and hence A has a reduced primary decomposition.

NOTE 4.5 : In an arbitrary Γ-semigroup it is not necessarily true that every Γ-ideal has a primary decomposition even if the Γ-semigroup is finite.

EXAMPLE 4.6 : Let S = {a, b, c} and Γ = {x, y, z}. Define a binary operation **.** in S as shown in the following table.

.	a	b	c
a	a	a	a
b	a	a	a
c	a	b	c

Define a mapping $S \times \Gamma \times S \longrightarrow S$ by $a\alpha b = ab$, for all $a, b \in S$ and $\alpha \in \Gamma$.

It is easy to see that S is a Γ-semigroup. Now consider the Γ-ideal $< a > = S^1\Gamma a\Gamma S^1 = \{a\}$.

Let $p\Gamma q \subseteq < a >$, $p \notin < a > \Rightarrow q \in \sqrt{< a >} \Rightarrow (q\Gamma)^{n-1}q \subseteq < a >$ for some $n \in N$.

Since $b\Gamma c \subseteq < a >$, $c \notin < a > \Rightarrow b \in < a >$. Therefore $< a >$ is left primary.

If $b \notin < a >$ then $(c\Gamma)^{n-1}c \notin < a >$ for any $n \in N \Rightarrow c \notin \sqrt{< a >}$.

Therefore $< a >$ is not right primary. In the Γ-semigroup S, $\{ b, c \}$ and $\{ a, b, c \}$ are the only primary Γ-ideals and hence $\{ a \}$ has no primary decomposition.

DEFINITION 4.7 : A Γ-semigroup S is said to be a ***noetherian Γ-semigroup*** if ascending chain if Γ-ideals becomes stationary.
i.e., if $A_1 \subseteq A_2 \subseteq A_3 \subseteq \cdots$ is an ascending chain of Γ-ideals of S, then there exists a natural number *m* such that $A_m = A_n$ for all natural numbers $n \geq m$.

THEOREM 4.8 : Every Γ-ideal in a (left, right) duo noetherian Γ-semigroup S has a reduced (right, left) primary decomposition.

***Proof* :** Let Σ be the collection of all Γ-ideals in S which has no primary decomposition. If Σ is not empty, then since S is noetherian, Σ contains maximal elements. Let C be a maximal element in Σ. Clearly C is not primary. Suppose that C is not left primary. Then there exists elements a, b in S such that $< a > \Gamma < b > \subseteq C$, $b \notin C$ and $a \notin \sqrt{C}$. Since S is a duo Γ-semigroup and hence by theorem 2.39, $\sqrt{C} = \{x \in S : (x\,\Gamma)^{n-1}x \subseteq C$ for some natural number $n\}$. Therefore $(a\Gamma)^{n-1}a \nsubseteq C$ and hence $(a\gamma)^{n-1}a \notin C$ for some $\gamma \in \Gamma$. For any natural number *n*, write $B_n = \{x \in S : (a\gamma)^n x \in C\}$. Let $x \in B_n$ and $s \in S$. $x \in B_n \Rightarrow (a\gamma)^n x \in C$. $(a\gamma)^n x \in C$, $s \in S \Rightarrow (a\gamma)^n x\gamma s \in C \Rightarrow x\gamma s \in B_n$. Therefore B_n is a right Γ-ideal in S. Since S is duo Γ-semigroup, B_n is a Γ-ideal in S. Now $B_1 \subseteq B_2 \subseteq \cdots$ is an ascending chain of Γ-ideals in S. Since S is noetherian there is a natural number k such that $B_k = B_i$ for all $i \geq k$. Since $b \in B_k$, we have B_k contains C properly. Write $D = (a\gamma)^k S \cup C$. Since S is a duo Γ-semigroup, D is a Γ-ideal in S and containing C properly. Now we prove that $C = B_k \cap D$. Clearly $C \subseteq B_k \cap D$. If $x \in B_k \cap D$ and $x \notin C$, then $x = (a\gamma)^k y$ for some $y \in S$. Since $x \in B_k$, we have $(a\gamma)^k x \in C$. Therefore $(a\gamma)^{2k} y = (a\gamma)^k (a\gamma)^k y = (a\gamma)^k x \in C$. Therefore $(a\gamma)^{2k} y \in C$. So $y \in B_{2k} = B_k$. Thus $x = (a\gamma)^k y \in C \Rightarrow x \in C$. It is a contradiction. So $B_k \cap D \subseteq C$ and hence $C = B_k \cap D$. Since B_k and D contains C properly and C is maximal in Σ, B_k and D have primary decompositions and hence C has a primary decomposition. It is a contradiction. Thus C is left primary. Similarily we can prove that C is right primary. Hence C is primary. It is a condiction. Therefore Σ is empty. Thus every Γ-ideal in a duo noetherian Γ-semigroup has a primary decomposition and hence by theorem 4.4, every Γ-ideal has a reduced primary decomposition.

COROLLARY 4.9 : Every Γ-ideal in a commutative noetherian Γ-semigroup S has a reduced primary decomposition.

THEOREM 4.10 : Let A and B be two Γ-ideals in a Γ-semigroup S. Then A^l (B) = { x ∈ S : < x >ΓB ⊆ A} is a Γ-ideal of S containing A.

Proof **:** Let $x \in A^l(B)$, $s \in S$ and $\gamma \in \Gamma$.
$x \in A^l(B) \Rightarrow \langle x \rangle\Gamma B \subseteq A$. Now $\langle s\gamma x \rangle\Gamma B \subseteq \langle x \rangle\Gamma B \subseteq A \Rightarrow s\gamma x \in A^l(B)$.

And $\langle x\gamma s \rangle\Gamma B \subseteq \langle x \rangle\Gamma B \subseteq A \Rightarrow x\gamma s \in A^l(B)$.

Therefore $s\gamma x, x\gamma s \in A^l(B)$. Hence $A^l(B)$ is a Γ-ideal of S containing A.

THEOREM 4.11 : Let A and B be two Γ-ideals in a Γ-semigroup S. Then A^r (B) = { x ∈ S : B Γ< x > ⊆ A} is a Γ-ideal of S containing A.

Proof **:** Let $x \in A^r(B)$, $s \in S$ and $\gamma \in \Gamma$.
$x \in A^r(B) \Rightarrow B\Gamma\langle x \rangle \subseteq A$. Now $B\Gamma\langle s\gamma x \rangle \subseteq B\Gamma\langle x \rangle \subseteq A \Rightarrow s\gamma x \in A^r(B)$.

And $B\Gamma\langle x\gamma s \rangle \subseteq B\Gamma\langle x \rangle \subseteq A \Rightarrow x\gamma s \in A^r(B)$.

Therefore $s\gamma x, x\gamma s \in A^r(B)$. Hence $A^r(B)$ is a Γ-ideal of S containing A.

THEOREM 4.12 : If A is a left primary Γ-ideal of a Γ-semigroup S, then A^l (B) is a left primary Γ-ideal.

Proof **:** If B ⊆ A, then clearly $A^l(B) = S$. Suppose $B \nsubseteq A$. Let $b \in B\backslash A$. Let $x \in A^l(B)$.
Then $\langle x \rangle \Gamma B \subseteq A$. So $\langle x \rangle \Gamma \langle b \rangle \subseteq A$. Since $b \notin A$, We have $x \in \sqrt{A}$ and hence

$\sqrt{(A^l(B))} = \sqrt{A}$. Let $\langle x \rangle\Gamma\langle y \rangle \subseteq A^l(B)$ and $y \notin A^l(B)$. Now $\langle x \rangle\Gamma\langle y \rangle\Gamma B \subseteq A$.

If $x \notin \sqrt{(A^l(B))} = \sqrt{A}$, then $\langle y \rangle \Gamma B \subseteq A$ and hence $y \in A^l(B)$. It is a contradiction.

So $x \in \sqrt{(A^l(B))}$. Therefore $A^l(B)$ is a left primary Γ-ideal.

THEOREM 4.13 : If A is a right primary Γ-ideal of a Γ-semigroup S, then A^r (B) is a right primary Γ-ideal.

Proof **:** If B ⊆ A, then clearly $A^r(B) = S$. Suppose $B \nsubseteq A$. Let $b \in B\backslash A$. Let $x \in A^r(B)$.
Then $B\Gamma\langle x \rangle \subseteq A$. So $\langle x \rangle \Gamma \langle b \rangle \subseteq A$. Since $b \notin A$, We have $x \in \sqrt{A}$ and hence

$\sqrt{(A^r(B))} = \sqrt{A}$. Let $\langle x \rangle\Gamma\langle y \rangle \subseteq A^r(B)$ and $x \notin A^r(B)$. Now $\langle y \rangle\Gamma\langle x \rangle \Gamma B \subseteq A$.

If $y \notin \sqrt{(A^r(B))} = \sqrt{A}$, then $\langle x \rangle \Gamma B \subseteq A$ and hence $x \in A^r(B)$, a contradiction.

So $y \in \sqrt{(A^r(B))}$. Therefore $A^r(B)$ is a right primary Γ-ideal.

THEOREM 4.14 : If Q is a P-primary Γ-ideal and if A ⊈ P, then Q^l (A) = Q^r (A) = Q and also if A ⊆ P and A ⊈ Q, then $\sqrt{(Q^l(A))} = \sqrt{(Q^r(A))} = \sqrt{Q}$.

Proof **:** Clearly $Q \subseteq Q^l(A)$. Let $x \in Q^l(A)$. Then $\langle x \rangle\Gamma A \subseteq Q$. Since $A \nsubseteq P$, there exists $a \in A\backslash P$. Now $\langle x \rangle\Gamma A \subseteq Q$ and $a \notin \sqrt{Q}$. So $x \in Q$. Therefore $Q^l(A) = Q$.
Similarly we can show that $Q^r(A) = Q$. The proof of the second part is evident.

THEOREM 4.15 : If $A_1, A_2, \ldots, A_n$ B are Γ-ideals of a Γ-semigroup S, then $\left(\bigcap_{i=1}^{n} A_i\right)^l (B) = \bigcap_{i=1}^{n} (A_i)^l (B)$.

Proof **:** $x \in \left(\bigcap_{i=1}^{n} A_i\right)^l (B) \Leftrightarrow \langle x \rangle \Gamma B \subseteq \cap A_i \Leftrightarrow \langle x \rangle \Gamma B \subseteq A_i$ for $i = 1, 2, 3\ldots, n$.

$\Leftrightarrow x \in A_i^l(B)$ for $i = 1, 2, 3\ldots, n \Leftrightarrow x \in \bigcap_{i=1}^{n} A_i(B)$. Similarly we can show that if $x \in \cap A_i^l(B)$.

Then $x \in (\cap A_i)^l(B)$. Therefore $\left(\bigcap_{i=1}^{n} A_i\right)^l (B) = \bigcap_{i=1}^{n} (A_i)^l (B)$.

THEOREM 4.16 : Suppose a Γ-ideal A in a Γ-semigroup S has two reduced (one sided) primary decompositions $A = A_1 \cap A_2 \cap \ldots \cap A_k = B_1 \cap B_2 \cap \ldots B_s$, where A_i is P_i -primary and B_j is Q_j-primary. Then k = s and after reindexing if necessary $P_i = Q_i$ for *i* = 1, 2, …,k. Further if each P_i is an isolated prime, then $A_i = B_i$ for *i* = 1, 2, …., *n*.

Proof **:** Let P_k be the maximal lement in the set $P_1, P_2, \ldots, P_k, Q_1, Q_2, \ldots, Q_s$.
Now we show that P_k occurs among $Q_1, Q_2, \ldots, Q_s$.

For this it is enough to show that $P_k \subseteq Q_j$ for some j. If $A_k \subseteq Q_j$ for some j, $P_k = \sqrt{A_k} \subseteq Q_j$.

Suppose $A_k \nsubseteq Q_j$ for all j. Then by theorem 4.12, $B_j^l (A_k) = B_j$ for all j.

Now $A^l(A_k) = (B_1 \cap B_2 \cap \ldots \cap B_s)^l(A_k)$

$= B_1^l(A_k) \cap B_2^l(A_k) \cap \ldots \cap B_s^l(A_k)$, by using theorem 4.12,

$= B_1 \cap B_2 \cap \ldots \cap B_s = A$.

But on the other hand if $1 \leq i < k$, then $P_k \nsubseteq P_i$ and therefore $A_k \nsubseteq P_i$, so that $A_i^l(A_k) = A_i$ and $A_k^l(A_k) = S$.

So we have $A^l(A_k) = (A_1 \cap A_2 \cap \ldots \cap A_k)^l(A_k) = A_1^l(A_k) \cap A_2^l(A_k) \cap \ldots \cap A_k^l(A_k)$

$= A_1 \cap A_2 \cap \ldots \cap A_{k-1}$. Therefore $A = A_1 \cap A_2 \cap \ldots \cap A_{k-1}$.

It is a contradiction to the fact that given decomposition is reduced.

Thus $A_k \subseteq Q_j$ for some j and hence $P_k \subseteq Q_j$. Therefore $P_k = Q_j$.

Without loss of generality we may assume that $P_k = Q_s$.

Let $B = A_k \cap B_s$. By theorem 4.2, B is a primary Γ-ideal and $P_k = Q_s$(= P say) is a prime Γ-ideal belonging to B. Since $P \nsubseteq P_i$ for all *i* , $1 \leq i < k$ and $B \subseteq A_k$, we have $A_i^l(B) = A_i$ and $A_k^l(B) = S$.

Therefore $A^l (B) = A_1 \cap A_2 \cap \ldots \cap A_{k-1}$.

Similarly we can show that $A^l(B) = B_1 \cap B_2 \cap \ldots \cap B_{s-1}$.

Hence $A^l(B) = A_1 \cap A_2 \cap \ldots \cap A_{k-1} = B_1 \cap B_2 \cap \ldots\ldots \cap B_{s-1}$ are two reduced primary decompositions for $A^l(B)$.

By continuing the above process, we get $k = s$ and $P_i = Q_i$ for $i = 1, 2, \ldots, k$.

Suppose P_i's are isolated primes.

If $A_1 \nsubseteq B_1$ then since B_1 is primary and $A_1 \cap A_2 \cap \ldots\ldots \cap A_k \subseteq B_1 \cap B_2 \cap \ldots\ldots \cap B_k \subseteq B_1$, we have $A_2 \cap A_3 \cap \ldots\ldots \cap A_k \subseteq \sqrt{B_1} = P_i$.

Now $P_2 \cap P_3 \cap \ldots\ldots \cap P_k = \sqrt{A_1 \cap A_2 \cap \ldots . \cap A_k} = P_1$.

Since P_1 is a prime Γ-ideal, $P_i \subseteq P_1$ for some $1 < i \leq k$.

It is a contradiction to the fact that P_1 is an isolated prime.

So $A_1 \subseteq B_1$. Similarly we can show that $B_1 \subseteq A_1$. Therefore $A_1 = B_1$.

By continuing in this way we get $A_i = B_i$ for some $i = 1, 2, \ldots\ldots, k$.

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