INTERNATIONAL JOURNAL OF MATHEMATICAL SCIENCES, TECHNOLOGY AND HUMANITIES

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International eJournals

International Journal of Mathematical Sciences, Technology and Humanities 49 (2012) 498 – 519

Fixed Point Theorems in 2 – Uniform Space

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1. INTRODUCTION:

In this paper we have introduced contraction type mappings in 2 – Uniform spaces and them some fixed point theore as have been proved in 2 – uniform space. Our results generalizes the results of many authors such as Lal and Singh [3], Das and Sharms [1] Singh and Singh [5] etc.

1.1 PRELIMINARIES:

In this section we shall do some definitions and lemmas.

1.1 **DEFINITION: A Pseudo – 2 – Metric** p for a set X in a real valued function defined on X x X x X, such that for all a,b,c,d, € X, we have
(i) p(a,b,c) **>** O and p(a,b,c) .. O. If at least two of a.b.c are equal.
(ii) p(a,b,c) = p(b,c,a) = p(c,a,b) = ….. so on.
(iii) $p(a,b,c) \leq p(a,b,d) + p(a,d,c) + p(d,b,c)$.

A set X together with a pseudo 2 - metric P Is called pseudo – 2 – metric space (X,p).

(1.1.2) **DEFINITION:** A 2 – uniformity for a set X is a non void family .. of subsets of X x X x X such that

(u_1) each member of …. Contains the diagonal Δ of X^3, Δ = [(x,x,x) : x ε X}
(u_2) If u ε … then vo vo v $\subseteq$ u for some v in ….
(u_3) If u and v are members of … then u $\cap$ v ε …..
(u_4) If u ε ………… and u $\subseteq$ v $\subseteq$ X^3 then v ε …….

By 2- uniform space, we mean a set X endowed with 2 – uniformity … in X, written as (x,….)

(1.1.3) **EXAMPLE:** Every 2 – metric space (X,d) is 2 – uniform space.

(1.1.4) **DEFINITION:** If (x, ….) is a 2 – uniform space, then a subject …. Of … will be called a basis for (x, ….)

(i) if x ε X and u ε …… , then (x,x,x) ε ….
(ii) if u ε …, then u^{-1} contains a member of ……
(iii) If u ε ………, then vo vo v $\subseteq$ u for some v in ………
(iv) for each u ε …. And v ε …… there is a w ε ……….. in which w $\subset$ u $\cap$ v.

(7.1.5) DEFINITION: A net ϕ : D $\rightarrow$ X in a space X is said to converge to a point x ε X iff ϕ is eventually in every neighbourhood of p.

DEFINITION: By Cauchy net (or fundamental net) in a 2 – uniform space (X, ……), we mean a net ϕ : D $\rightarrow$ X in the space X such that for an arbitrary member u of there exists a residual subset B of D satisfying (ϕ (a), ϕ (b), c) ε …… , for any three members a,b and c of E.

(7.1.6) DEFINITION: $\wedge$ 2 – uniform space (x, ……) is called
Sequentially complete if every Cauchy sequence in X converge to a point in X.

Now, for any pseudo – 2 – metric p on any r > O, we write

$V_{(p,r)}$ = { (x,y,z) : x,y,z ε X and p(x,y,z) < r }
Let P be a family of pseudo – 2 – metrics on X

Generating the uniformity. Denote V the family of all
Sets of the form $\bigcap_{1=1}^{n}$ V (p_1, r_1), where P_1 ε P and

r_1 $\triangleright$ O, 1 = 1,2 ………. N (the integer is not fixed). Then clearly V is a base for the uniformity …….
Let V ε V, then v = $\bigcap_{1=1}^{n}$ V (p_1, r_1), where
P_1 ε P and r_1 > O, 1 =1,2 ……………….. n, For each ε > O_1,

The set $\bigcap_{1=1}^{n}$ V(P_1,α r_1), belongs to V we denote this set by α v.

(1.1.8) **LEMMA:** If v ε V and α, β are positive
Then α(β v) = (a β) v.

(1.1.9) **LEMMA:** If v ε V and α, β are positive
Then α v ⊂ β v where α < β.

(1.1.10) **LEMMA:** Let p be any pseudo - … - metric on Ä
And α,p be any two positive numbers, If (x,y,z)
ε α v (p,r_1) o β v (p, r_2) then
P(x,y,z) < αr_1 + β r_2.

(1.1.11) **LEMMA**: If v ε V and α, β are positive,
Then αv o β v ⊂ (α + β) v.

(1.1.12) **NOTE:** Let p be any pseudo – 2 – metric an x and α, β,γ be three positive numbers.

If (x,y,z) ε α v (p,r_1) $^{o\beta v}$ (p,r_2)$^{o\gamma v}$ (p,r_3).
Than p(x,y,z) < α r_1 + βr_2 + γ r_3

(1.1.13) **LEMMA:** Let x,y,z ε X, then for every v in v there is a positive number λ such that (x,y,z) ε λ v. The proofs of …. 1.1.8 – 1.1.13 are simple hence we omit here.

(1.1.14) **LEMMA:** Let v be any member of V. Then there is a pseudo – 2 – metric p on X, s,t. v = V(p,l).

Proof: Let (x,y,z) be any three points of X, The by lemma (1.1.15) there is a λ > ∪ such that (x,y,z) ε λ , ∀ write $A_{(x,y,z)}$ = { λ : λ > ∪ and (x,y,z) ε λ , ∀ }

Now we define p(x,y,z) by p(x,y,z) = Inf {λ : λ ε A (x,y,z) }.

If x ε X, then clearly (X x X x X) ε v for any > o.
This shows that A(x,x,x) = { λ : λ > o }.

So p(x,x,x) = Inf A(x,x,x) = O. Again since v is symmetric it follows that

A(x,y,z) = A (y, z, x) = A(z, x, y) = ………………….

So,

P(x,y,z) = p(z,x,y) = p(y,z,x) = …………. ≥ O.
Now Let x,y,z,a be any four points of X. Choose ε > O
Arbitrarily, Tak α = p (x,y,a) + ε,

β = p ø(x,a, z) > ε and γ = p (a, y, z) + ε

The, α ε A (x,y,a), β ε A (x, a,z) and γ ε A (a,y,z)
i.e (x,y,a) ε α v, (……….) ε β ν and (a,y,z) ε γ v,

This gives that (x,y,z) ε β ν o α v o γ v = α v o β v o γ v ⊂ (α+β+ γ) v (by note 1.1.12)

Thus, (α+β+γ) ε A (x,y,z).
So,

P(x,y,z) is α + β + γ = p (x,y,a) + p(x,a,z) + p(a,y,z) + 3 ε
Since ε > ∪ is arbitrarily, be get

P(x,y,z) < p(x,y,a) + p(x,a,z) + p(a,y,z)
Thus, p is a pseudo - 2 metric on X.

Let x,y,z ε $\dddot{\wedge}$ and p(x,y,z) < 1. Choose any α with
p (x,y,z) < α < 1, Then α ε A (x,y,z) which gives that

(x,y,z) ε α v ⊂v. (By lemma 1.1.9). So
(I) $v_{(p,1)} \subset V$

Again let (x,y,z) ε V. Since v ε V. We can choose
$V = \bigcap_{1=1}^{n} v_{(p1,\, r1)}$, P_1 ε P and $r_1 > O$.

Write $\alpha_1 = P_1$ (x,y,z), then $O \le a_1/r_1$ x 1, (1 = 1,2,….n)

Let θ = max, $\left\{\frac{a_1}{r_1}, 1 = 1, 2 \ldots\ldots\ldots n\right\}$. Then O ≤ θ ≤ 1.

Choose any positive a with θ < α < 1, we have

P_1 (x,y,z) = $\alpha_1 = \frac{a_1}{r_1} r_1 \le \theta\, r_1 \le r_1$ (1 = 1,2 …………….., n)

So, $(x,y,z) \varepsilon \bigcap_{1=1}^{n} V(p_1,r_1) = \alpha \nu$

And hence $p(x,y,z) \le \alpha \le 1$. Thus
(II) $V \subset V_{(p,1)}$,

From I and II, we get V = V(p,l).

NOTE: We shall call p- Minkowski's pseudo – 2 – Metric of V.

(1.1.15) DEFINITION: Let β be a basic for the 2 – uniform Space(X,…..) and let f be a function on X into X, then

(a) f is said to be a contraction with respect to ……
If (f(x), f(y), z) ε U whenever (x,y,z) ε U ε ….

(b) f is said to be expansion with respect to …..
If (x,y,z) ε U whenever (f(x), f(y), z) ε U ε …….

1.2 RESULTS OF FIXED POINT OF OPERATIONS:

In this section we assure that (x, ……) is a 2 – uniform space which is sequentially complete and also Hausdorif, Further we suppose that P is a fixed family of paseudo – 2 – metric on X which generates the uniformity ….We denote γ the family of all sets the form $\bigcap_{1=1}^{n} V(p_1, r_1)$, P_1 ε P and $r_1 > O$.

(the integer n is not fixed).

By an operator on X we mean a mapping of X into itself.

(1.2.1) THEOREM: Let $\{S_1, S_2, \ldots S_{q_1}\}$ and $\{T_1, T_2 \ldots T_{q_2}\}$ Be two sets of operators such that

(i) S_1 $(1 \le 1 \le q_1)$ and T_μ $(1 \le \mu \le q_2)$ all maps X into itself.
(ii) $T_\mu T_\gamma = T_\gamma T_\mu$ where $1 \le \mu \gamma \le q_2$.
(iii) For all x, y, ε X and for every α ε X and each p ε P , any five members V_1, V_2, V_3, V_4, V_5 in V
$(S(x), T(y), \alpha))$ ε $\alpha_1 v_1$ o $\alpha_2 \nu_2$ o $\alpha_3 v_3$ o $\alpha_4 \nu_4$, o $\alpha_5 v_5$ with
$S = S_1 \ldots S_{q_1}$; $T = T_1 \ldots T_{q_2}$. If
(x, S(x), a) ε V_1, (y, T(x), a) ε V_2, (x, T(y), a) ε V_3

(y, S(x), a) ε V_4, (x,y,α) ε V_5, where each

α_1 (1 =1, 2,……….. 5) are non-negative real numbers
Independent of x,y,α, V_1, V_2, V_3, V_4,V_5 such that

(iv) $O < \frac{a_1 + a_3 a_5}{1 - a_2 - a_3} : \frac{a_2 + a_4 + a_5}{1 - a_2 - a_3} < 1, 1 - \alpha_2 - \alpha_3 \neq o,\ 1 - \alpha_1 - \alpha_4 \neq o.$

Then S_1 (1 < I < q_1) and T_μ (1 < μ < q_2) all have a unique common fixed point.

Proof: From given condition (iv) we have
$\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 < 1$ ………………….. (1)

Suppose $K_1 = \frac{a_1 + a_3 + a_5}{1 - a_2 - a_3} : K_2 = \frac{a_2 + a_4 + a_5}{1 - a_1 - a_4}$.....(2)

Let v be any member of V and p be the Mindowski's
2 pseudo 2 – retric of V. Consider x,y,a be any three

Points of X.
Put, p(x, S9x), a) = r_1 : p(y, T(y), a) = r_2,
P(x, T(y), a) = r_3 : P(y, S(x), a) = r_4',
P (x,y,a) = r_5 and take ε > O, then
(x, S(x), a) ε ($x_1 + \varepsilon$) v, (y, T(y), a) ε ($r_2 + \varepsilon$) V,
(x, T(y), a) ε ($r_3 + \varepsilon$)v, (y, S(x), a) ε ($r_4 + \varepsilon$) v,
(x,y,a) ε ($r_5 + \varepsilon$) v. Then by given condition we have
(S(x), T(y), a) $\in a_1 (r_1 + \varepsilon)$ v o $a_2 (r_2 + \varepsilon)$ v o $a_3 (r_3 + \varepsilon)$ v o $a_4 (r_4 + \varepsilon)$ v o $a_3(r_5 + \varepsilon)$ v.

Then by lemma (1.1.10), we get

$P(S(x), T(y), a) < a_1(r_1 + \varepsilon) + a_2(r_2 + \varepsilon) + a_3 (r_3 + \varepsilon) + a_4(r_4 + \varepsilon) + a_3 (r_5 + \varepsilon)$
$= a_1 r_{1= +} a_2 r_2 + a_3 r_3 + a_4 r_4 + a_5 r_5 + (a_1 + a_2 + a_4 + a_5) \varepsilon$

As ε is arbitrary, we have

$p(S(x), T(y), a) \leq a_1 p(x, S(x), a) + a_2 p (y, T(y), a) + a_3 p (x, T(y),a) + a_4 P(y, S(x),a) + a_5 p (x,y,a).$

We take any X_O ε X and construct a sequence {x_h } in X by setting

$X_{2n+1} = S(x_{2', n})$ and $x_{2n+2+} = Tx_{2n+}$ for n = o,1,2,………………..(3).

Now, $p(x_{2n-1,} x_{2n'} a) = p(S(x_{2n-2)}, T(x_{2n-1}), a)$

$< a_1 P (X_{2n-2}), T(x_{2n-2}), a)$
$+ a_2 p(x_{2n-1,} T(x_{2n-1}), a)$
$+ a_{3,} p(x_{2n-2,} T(x_{2n-1}), a)$
$+ a_4 \; p (x_{2n-1,} T(x_{2n-1}), a)$
$+ a_5 p(x_{2n-2,} T(x_{2n-1}), a)$

Hence, $P(x_{2n-1}, x_{2n}, a) \ldots\ldots k_1 p(x_{2n-2,} T(x_{2n-1}), a)$

From (2) ……………………. (4)

Now, $p(x_{2n-1,} x_{2n+1}, a) = p(T(x_{2n-1}), S(x_{2n})_, a)$

$= p(S(x_{2n})_, T(x_{2n-1}),a)$
$\leq a_1 P (X_{2n,} S x_{2n,} a)$
$+ a_2 p(x_{2n-1,} Tx_{2n-1},a)$
$+ a_3 p(x_{2n,} Tx_{2n-1},a)$
$+ a_4 p(x_{2n-1,} Sx_{2n},a)$
$+ a_5 p(x_{2n-1,} x_{2n},a)$

i.e $p(x_{2n,} x_{2n-1},a) \leq k_2 p(x_{2n-1,} x_{2n},a)$ ………………………….. (5)
$\leq k_1 k_2 p(x_{2n-2,} x_{2n-1},a)$
$\vdots$
$\leq k_1^n k_2^n p(x_{o,}x_1,a)$ …………….. (6)

and $p(x_{2n+1,} x_{2n+2},a) \leq k_1 p (x_{2n,} x_{2n+1,}a)$
$\vdots k_1 k_1^n k_2^n p(x_{o,}x_1,a)$ (from (6))
$\frac{2_n+1}{2}$ ……………..(7)

$= (1 + k_1) (k_1 k_2) \quad p(x_o,x_1,a)$

Therefore by repeated use of trainable inequality and of Reduction formulas (6) and (7),

we get

$P(X_m, X_{m+n}, a) \leq p(X_m,X_{m+1}, X_{m+n}) + p(X_m, X_{m+1},a)$
$+ p(X_{m+1,} X_{m+2,} X_{m+n)} + p(X_{m+1,} X_{m+2,}a)$
$+$ …………………….. $+$ ……………………
$+ p(X_{m+n-2,} X_{m+n-1,} X_{m+n}) + p(X_{m+n-1,} X_{m+n,}a)$

Now, $p(x_{2n-1,} x_{2n+1}, a)= p(T(x_{2n-1}), S(x_{2n}), a)$

$= p(S(x_{2n}), T(x_{2n-1}), a)$
$\leq a_1\, p(x_{2n}, Sx_{2n}, a)$
$+ a_2\, p(x_{2n-1}, Tx_{2n-1}, a)$
$+ a_3\, p(x_{2n}, Tx_{2n-1}, a)$
$+ a_4\, p(x_{2n-1}, Sx_{2n-1}, a)$
$+ a_5\, p(x_{2n-1}, x_{2n}, a)$

i.e. $p(x_{2n}, x_{2n+1}, a) \leq K_2\, p(x_{2n-1}, x_{2n}, a)$ ………… (5)
$\leq k_1\, k_2\, p(x_{2n-2}, x_{2n-1}, a)$
:
$\leq$
:
$\leq k_1^{\,n} k_2^{\,n}\, p(x_0, x_1, a)$ ………….. (6)

And $p(x_{2n+2}, x_{2n+2}, a) \leq k_1\, p(x_{2n}, x_{2n+1}, a)$
:
$\leq$
:
:
:
:
$\leq k_1\, k_1^{\,n}\, k_2^{\,n}\, p(x_o, x_1, a)$ (from (6))

$$= (1 + k_1)\,(k_1\, k_2)^{\frac{2_n + 1}{2}}\; P(x_o, x_1, a) \quad \text{……………..(7)}$$

Therefore by repeated use of triangle inequality and of reduction formulas (6) and (7), we get

$P(x_m, x_{m+n}, a) \leq p\,(x_m, x_{m+1}, x_{m+n}) + p(x_m, x_{m+1}, a)$-
$+p(x_{m+1}, x_{m+2}\, x_{m+n}) + p(x_{m+1}, x_{m+2}, a)$
$+$ …………………. $+$ ………………..
$+\, p(x_{m+n-2}, x_{m+n-1}, x_{m+n}) + p(x_{m+n-1}, x_{m+n}, a)$

Now we show that $p(x_m, x_{m+1}, x_{m+2}) = O$.

$p(x_{m+1}, x_{m+2}, x_m) = p(S_{xm}, Tx_{m+1}, x_m)$
$\leq a_1 p\,(x_m, Sx_m, sx_m) + a_2(x_{m+1}, Tx_{m+1}, x_m)$
$+ a_3\, p(x_{m+1}, Tx_{m+1}, x_m)\; a_4 p\,(x_{m=1}, Sx_m, x_m)$
$+ a_5\, p(x_m, x_{m+1}, x_m)$
$= a_1 p(x_m, x_{m+1}, x_m) + a_2\, p(x_{m+1}, x_{m+2}, x_m)$
$+ a_3\, p\,(x_m, x_{m+1}, x_m) + a_4\, p(x_{m+1}, x_{m+1}, x_m)$

$+ a_5 \, p\,(x_{m,}\, x_{m+1},\, x_m) + a$
$= a_1 O + a_2 p\,(x_{m+1,}\, x_{m+2,}\, x_m) + a_3^{O} + a_4^{O} + a_5^{O}.$

$(1 - a_2) \quad p(x_{m+1,}\, x_{m+2,}\, x_m) \leq O$ which implies that $p\,(x_{m+1,}\, x_{m+2,}\, x_m) = O$.

Now, we show that $p(x_o, x_1, x_m) = O$ for m = O,1,2,………..

This is true for m = O, and m = 1, Suppose now that it holds for every m in $2 \leq m \leq k - 1$. Then

$$p(x_o, x_1, x_k) \leq p(x_o, x_1, x_{k-1}) + p(x_o, x_{k-1}, x_k) + p(x_{k-1,} x_1, x..)$$
$$\leq (1 + k_1)(k_1 k_2)\ \frac{k-1}{2}\ [\,p(x_o, x_o, x_1) + p(x_o, x_1, x_1)] = O$$

Hence $p\,((x_o, x_1, x_m) = O$

Since $p\,(x_m, sx_{m+1}, x_{m+n}) \leq (1+k_1)\,(k_1k_2)^{m/2}\,[p(x_o, x_1, x_{m+n})]$,

It follows that $p(x_m, x_{m+1}, x_{m+n}) = o$ and thus

$$P(x_m, x_{m+n}, a) \leq (1+k_1)\,(k_1 k)^{m/2} + (k_1 k_2) + \frac{m+1}{2} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$\ldots\ldots\ (k_1 k_2)\ \frac{m+n-2}{2}\ ]\ p(x_o, x_1, a)$$

As $k_1 k_2 < 1$ the R.H,S of the above inequality tends to zero as $n \to \infty$, Hence (x_n) is a Cauchy sequence. Since X is sequentially complete there is a point … in x such that $u = n \underset{\to}{L_t} \infty, X_n$

Now we show that u is a unique common fixed point of s and T. Let v be any member of V and p be the Minkowski's pseudo 2 – metric of v. For any positive integer n, we have

$$P(u, S_u, a) \leq p(u, S_{u,} x_{2n}) + p(u, x_{2n}, a) + p(x_{2n},, Su, a)$$
$$= p(u, {}_{,}x_{2n,}\, a) + p(u, Su, x_{2n}) + p(Tx_{2n-1}, Su, a)$$
$$\leq p(u, {}_{,}x_{2n,}\, a) + p(u, Su, x_{2n}) + a_1\, p(u, S_{u,} a)$$
$$+ a_2 P(x_{2n-1,}\, Tx_{2n-1,} a) + a_3\, p(u, Tx_{2n-1,} a)$$
$$+ a_4 P(x_{2n-1,}\, Su, a) + a_5\, p(u, x_{2n,} a)$$
$$\infty\, p(u, x_{2n,}\, a) + p(u, Su, x_{2n}) + a_1\, p(u, Su, a)$$
$$+ a_2\, p(x_{2n-1}, x_{2n+1,} a) + a_3\, p(u, x_{2n}, a)$$
$$+ a_4 p(x_{2n-1,}\, Su, a) + a_3\, p(u, x_{2n,}\, a)$$

When $n \to \infty$, as $x_{2n} \to u$, $x_{2n-1} \to u$, $x_{2n-1} \to u$.

And thus
$p(u, Su, a) = a_1\, p(u, Su, a) + a_4\,(u, Su, a)$

or, $(1-a_1 - a_4)$ v (u, Su, a) $\leq$ O
i.e. p(u, Su, a) = O, So (u, Su, a) ε v.

v being arbitrary and X being Hausdorix space.

We have u = Su. Similarly, u = Tu.

For the uniqueness of u, let $\bar{u} \neq$ u is also fixed point common to both S and T such that S ($\bar{u}$) = T($\bar{u}$) = $\bar{u}$ giving p(u, $\bar{u}$,a) = p(Su, T$\bar{u}$,a)

$\leq a_1$ p(u, Su, a) + a_2 p($\bar{u}$, T($\bar{u}$), a) + a_3 p(u,T, ($\bar{u}$), a) + a_4 p($\bar{u}$, S(u),a) + a_3 p(u, $\bar{u}$, a).

i.e p(u, $\bar{u}$,a) $\leq a_1$ p(u, u, a) + a_2 p($\bar{u}$,$\bar{u}$,a) + a_3 p(u, $\bar{u}$,a) + a_4 p ($\bar{u}$,u ,a) + a_5 p(u, $\bar{u}$,a) which gives p(u, $\bar{u}$,a) $\leq$ O and thus u = $\bar{u}$.

Now we shall show that u is the unique common fixed point of S_1 ($1 \leq 1 \leq q_1$) and Tμ ($1 \leq \mu \leq q_2$).

For S(u) = u and $S(S_1(u)) = S_1S(u) = S_1$ (u).

i.e. $S_1(u)$ = u by the uniqueness of u as the fixed points of S. Similarly T_μ (u) = u. Finally we shall show that u is the only fixed point common to

S_1 ($1 \leq 1 \leq q_1$) and …..$_\mu$ ($1 \leq \mu \leq q_2$). For if u^* were such a point such that u^* = u and $S_1(u^*) = T_\mu (u^*) = u^*$
Then p(u, u^*, a) ….. p (S_1 (u), T_μ(u^*), a)
…… p (S(u), T(u^*), a)
$\leq a_1$ p(u, Su, a) + a_2 p(u^*, T u^*, a) +
a_3 p(u, T u^*, a) + a_4 p(u^*, Su, a) +
a_5 p (u, u^*, a)

Which gives p (u, u^*,a) = O and so u = u^*. //

(1.2.2) <u>THEOREM:</u> Let T_1 and T_2 be two operators such that

(i) T_1 and T=2 map X into itself
(ii) $T_1T_2 = T_2T_1$
(iv) for all x,y,z_1, z_2 ε X and each a ε x and each

p ε p any six members v_1, v_2, v_3,v_4, v_5, v_6 in v

$(T_1(x), T_2(y),a) \leq a_1 v_1 \text{ o } a_2 v_2 \text{ o } a_3 v_3 \text{ o } a_4 v_4 \text{ o } a_5 v_5 \text{ o } a_6 v_6$
If $(x,T_1^k(z_1), a) \varepsilon v_1$; $(y, T_2^k(z_2),a) \varepsilon V_2$; $(x,T_1^k(z_2), a) \varepsilon v_3$;
$(y, T_1^k(z_1),a) \varepsilon v_4$; $(T_1^k(x_1), T_2^k(x_2), a) \varepsilon v_5$ and
$(x,y,a) \varepsilon v_6$ where a_1 (i = 1, 2 ……………6) all are

Independent of x,y,a, z_1, z_2 and v_1, v_2 …………….. v_6 with

$a_1 \leq$ O for each i = 1,2 ……………. 6 : $\sum_{i=1}^{6} a_1 < 1$, $k \geq 1$ (k is an positive integer)

Then T_1 and T_2 have a unique common fixed point.

<u>Proof:</u> Suppose v be any member of v and p the Minkowski's pseudo – 2 – metric of v – write.
$P(x,T_1^k(z_1), a) = r_1$: $p(y, T_2^k(z_2), a) = r_2$
$P(x, T_2^k(z_2), a) = r_3$: $p(y, T_1^k(z_1), a) = r_4$
$P(T_1^k (z_1), T_2^k (z_2), a) = r_5$: $p (x,y,a) = r_6$.

For any $\varepsilon >$ O we have

$(x,T_1^k(z_1), a) \varepsilon (r_1 + \varepsilon) v : (y, T_2^k (z_2),a) \varepsilon (r_2 + \varepsilon)v$
$(x, T_2^k(z_1), a) \varepsilon (r_3 + \varepsilon) v : (y, T_1^k (z_1),a) \varepsilon (r_4 + \varepsilon)v$
$(T_1^k(z_1), T_2^k(z_2), a) \varepsilon (r_5 + \varepsilon)v : (x,y,a) \varepsilon (r_6 + \varepsilon) v$.

Then by given conditions

$(f(x), g(y), a) \varepsilon a_1(r_1 + \varepsilon) v_o a_2(r_2 + \varepsilon) v \text{ o } a_3(r_3 + \varepsilon) v \text{ o } a_4 (r_4 + \varepsilon) v \text{ o } a_5 (r_5 + \varepsilon) v \text{ o } a_6(r_6 + \varepsilon) v$

They by lemma (1.1.10) and since ε is arbitrary, we have

$P(f(x), g(y), a) < a_1 p (x,f^k(x_1), a) + a_2 p(y,g^k(z_2)a) +$
$a_3 p (x,g^k(z_2),a) + a_4 p(y,f^k(z_1),a) +$
$a_5 p (f^k(z_1),g^k(z_2), a) + a_6 p(x,y,a)$ ………….. (1)

For arbitrary z and w in X put

$X = T_2^k(x)$, $y = T_1^k(w)$, $x_1 = w$, $x_2 = x$ in (1) we get
$P(T(T_2^k(x), T_2(T_1^K(w)),a) < a\, p(T_2^k(z), T_1^K(w),a)$ ……… (2)

(where $a = a_1 + a_2 + a_5 + a_6 < 1$)

Let $x_o \varepsilon X$ be arbitrary, Consider $\{x_n\}$ as follows:

$$x_n = \begin{cases} T_1(X_{n-1}) \text{ when n is odd} \\ T_2(X_{n-1}) \text{ when n is even} \end{cases}$$

In view of the given condition (ii) we observe that

$X_{2n} = T_1^{n} T_2^{n} (x_o)$: $x_{2n+1} = T_1^{n+1} T_2^{n} (x_o)$

Let n > t which gives no = £ + g for some positive integer q > 1.

$P(x_{2n}, x_{2n+1}, a) = p(T_1^{t+g} T_2^{t+g} (x_o), T_2^{t+g} T_1^{t+g-1}(x_o), a)$
$\leq a\, p(T_1^{t+g-1} T_2^{t+g} (x_o), T_2^{t+g-1} T_1^{t+g-1} (x_o), a)$
Using (2)
……………………………………………..
……………………………………………..
$\leq a^{q} p(T_1^{t} T_2^{t+g} (x_o), T_2^{t} T_1^{t+g-1} (x_o), a)$
$= a^{q} p(T_1^{t+g-1} T_2^{t} (x_o), T_2^{t+g} T_1^{t} (x_o), a)$
……………………………………….
……………………………………….
$\leq a^{2g} p(t_1^{t+1} T_2^{t} (x_o), T_2^{t} T_1^{t} (x_o), a)$

i.e $p(x_{2n}, x_{2n+1}, a) \leq a^{2n-2t} p(x_{2t}, x_{2t+1}, a)$

Taking m > n > t, and preoceeding similarly to the previous theorem we can show that $\{x_n\}$ is a Cauchy sequence in X, Since x is sequentially complete Housdorff space, there exits u ε x such that $u = \underset{n\to\infty}{Lt}\, x^{n}$.

For any odd positive integer h, we have

$P(u, T_2(u), a) \leq p(u, T_2(u), x_h) + p(u, x_h, a) + p(x_h, T_2 u, a)$

$= p(u, T_2(u), x_h) + p(u, x_h, a) + p(T_1(x_{h-1}, T_2(u), a)$

Taking $x = x_{h-1}$: $y = u$, $z_1 = T_2^{k}(x_{h-1})$, $Z_2 = T_1^{k}(x_{h-1})$ in (1)

And using the above inequality, we get

$P(u, T_2(u), a) \leq p(u, T_2(u), x_h) + p(u, x_h, a)$

$+ (a_1 + a_3) p(x_{h-1}, x_{h+2k-1}, a)$

$+ (a_2 + a_4)(u, x_{h+2k-1}, a) + a_6 p(x_{h-1}, u, a)$

When $h \to \infty$ x_h, x_{h-1}, x_{h+2h-2} all tends to u.

i.e. $p(u, T_2(u), a) \leq O$. Thus $p(u, T_2(u), a)$ ε v.

v being an arbitrary and x being Hausdorff space.

We have $u = T_2(u)$. Similarly we can show that $u = T_1(u)$.

Thus u is a common fixed point of T_1 and T_2. To show that the uniqueness of u. Let

$\overset{\infty}{u} \neq u$ is also a common fixed point of T_1 and T_2 such that

$$T_1 \left(\overset{\infty}{u}\right) = T_2 \left(\overset{\infty}{u}\right) = \overset{\infty}{u}.$$

For this we put $x = u = z_2$ and $y = \overset{\infty}{u} = z$ in (1) and we get the desired result. //

(7.2.3) <u>THEORM:</u> Let T_1 and T_2 be two operators such that

(i) T_1, T_2 maps X into itself

(ii) $T_1, T_2 = T_2 T_1$

(iii) For all x,y,z, z_2, z_3 in < and for each a ε X and each p ε P, any five members v_1, v_2, v_3, v_4, v_5 in v.

$(T_1(x), T_2(y), a)$ ε $a_1 v_1$ o $a_2 v_2$ o $a_3 v_3$ o $a_4 v_4$ o $a_5 v_5$

If $(x, T_1^k (z_1), a)$ ε v_1 : $(y, g^k(x_2), a)$ ε v_2 :
$(T_1(x), T_1^k(X_3), a)$ ε v_3 : $(T_2(y), T_1^k(z_3), a)$ ε V_4 : (x,y,a) ε V_5 where each a_1 (i=1,2,3,4,5)

are independent of x,y,a,x_1,x_2,x_3 and v_1 v_2 v_3 v_4 v_5 with each

$$a_1 \; (i=1,2,3,4,5) \geq O, \quad \sum_{i=1}^{5} a_1 < 1, \; k \geq 1 \text{ (k is an integer).}$$

Then T_1 and T_2 have a unique common fixed point in X.

<u>Proof:</u> Let v be any member of V and p the Minkowaki's pseudo 2-metric of v. Put
$P(x, T_1^k (x_1), a) = r_1 : p(y, g\,k\,(x_2), a) = r_2 : p(T_1(x), T_1^k (z_3), a) = r_3$
$P(T_2(y), T_1^k(z_3), a) = r_4 : p(x,y,a) = v_5$. For any $\varepsilon > o$,

We have, $(x, T_1^k (x_1), a)$ ε $(r_1 + \varepsilon)$ v : $(y, T_2^k (z_2), a)$ ε $(r_2 + \varepsilon)$ v

$(T_1(x), T_2^k(x_3), a)$ ε $(r_3 + ε)$ v : $(T_1(y), T_1^k(z_3),a)$ ε $(r_4 + ε)$ v (x,y,a) ε $(r_5 + ε)$ v. Then by the given condition

$(T_1 (x), T_2(y),a)$ ε $a_1 (r_1 + ε)$ v o $a_2 (r_2 + ε)$ v o $a_3 (r_3 + ε)$ v o $a_4 (r_4 + ε)$ v o $a_5 (r_5 + ε)$ v.
Then by lemma (1.1.10) and since ε is arbitrary, thus

$$
\begin{aligned}
P(T_1(x)\ T_2(y), a) \le\ & a_1\ p(x,T_1^k(k), a) + a_2\ p(y, T_2^k(y), a) \\
& + a_3\ p\ (T_1(x), T_2^k(z_3), a) \\
& + a_4\ p(T_2\ (y), T_1^k(z_3), a) \\
& + a_5\ p(x,y,a)\ \ldots\ldots\ldots\ldots\ldots(1)
\end{aligned}
$$

For arbitrary z,w ε x, Put $x = T_2^k (z)$, $y = T_1^k(w)$,

$X_1 = w$, z_2 %, $z_3 = g\ (w)$ in (1) we get

$$
\begin{aligned}
P(T_1(T_2^k(\%)), T_2(T_1^k(w),a) \le\ & a_1\ p(T_2^k(z), T_1^k(w),a) \\
& +a_2\ p(T_1^k(w), T_2^k(z), a) \\
& + a_3\ p\ (T_1^k(T_2^k(z), T_1^k(T_2(w)), a) \\
& + a_4\ p\ (T_2T_1^k(w), T_1^k(T_2(w)), a) \\
& + a_5\ p\ (T_2^k(z), T_1^k(w), a)
\end{aligned}
$$

i.e. $p(T_1,T_2^k(z), T_2\ T_1^k(w), a) \le a\ p(T_2^k 9z), T_1^k(w),a)$ ………………..(2)

Where $a = \dfrac{a_2 + a_2 + a_5}{1 - a_3} < 1$

Let x_0 ε X be arbitrary, We define a sequence $\{x_n\}$ as foolows

$$
x_n = \begin{cases} T_1\ (X_{n-1}) \text{ If n is odd} \\ T_2\ (X_{n-1}) \text{ If n is even} \end{cases}
$$

Now as proved in earlier theorems we can show that $\{x_n\}$ is a Cauchy sequence and since x is sequentially complete Hausdorff space there exists a point u ε X such that $u = \underset{n \to \infty}{Lt}\ x_n$. For any positive integer h, we have

$$
\begin{aligned}
P(u, T_2(u),a) &\le p(u, T_2(u), x_n) + p(u, x_h, a) + p(x_h, T_2(u), a) \\
&= p(u, T_2(u), X_h) + p(u,x_h, a) + p\ (T_1\ (X_{h-1}), T_2(u), a)
\end{aligned}
$$

Taking $x = x_{h-1,}$ $y = u$, $x_1 = T_2^k(x_{h-1})$, $Z_2 = T_1^k(x_{h-1})$ in (1) and using the above inequality we get

$$
\begin{aligned}
P(u, T_2(u)a) \le\ & p(u,T_2(u), x_h) + p(u, x_h, a) + \\
& a_1\ p\ (x_{h-1,}\ T_1^K\ T_2^K\ (x_{n-1}), a + \\
& a_2\ p(u, T_2^k\ T_1^k\ (x_{n-1}), a +
\end{aligned}
$$

a_3 p(T_1, x_{n-1}, $T_1^k T_2^k(x_{n-1})$, a) +
a_4 p (T_2 (u), T_1^k $T_2^k(x_{n-1})$, a) +
a_5 p($x_{n-1,}$ u, a)
$\leq$ p(u, T_2(u), x_h) + p(u, x_h, a) + a_1 p (x_{h-1}, $x_{h+2\,k-1,}$ a)
+ a_2 p ($x_{h-1,}$ $x_{h+2k-1,}$a) + a_3p(x_h, $x_{h+2k-1,}$a)
+ x_4 p(T_2(u),$x_{h+2k-1,}$a) + a_5p($x_{h-1,}$u,a)

When h $\rightarrow \infty$, x_h, x_{h-1}, x_{h+2k-1} all tends to u.

Thus, p(u,T_2(u),a) $\leq$ p(u,T_2(u),u) + p(u, u, s) + (a_1 + a_2 + a_3 + a_5) p(u,u,a) + a_4p(T_2(u), u, a)
i.e. (1-a_4) p(u,T_2(u),a) $\leq$ O which gives

p(u, T_2(u), a) = O, Hence (u, T_2(u), a) ε v.

As v being arbitrary and X being a Hausdorff speace, we have u = T_2(u). Similarly u = T_1(u). Thus, u is a common fixed point of T_1 and T_2. To prove that u is the unique common fixed point of T_1 and $T_{2.}$ To prove that u is the unique common fixed point of T_1 and T_2. Let $u_o \neq$ u be another point such that $T_1(u_o)$ = $T_2(u_o)$ = u_o giving p(u, u_o, a) = p(T_1(u), $T_2(u_o)$, a). Taking x = u = z_2 and y = u_o = z_1 = z_3 in (1) we get the desired result.

(7.2.4) THEOREM: Let f and g be two operators such that

(i) f,g maps X into itself
(ii) f, g = g.f
(iii) for all x,y,x_1, x_2, z_3, z_4 ε X and for each a ε x and each p ε p any four members $v_{1,}$ $v_{2,}$ $v_{3,}$ $v_{4,}$ in v (f(x), g(y), a) ε a_1 v_1 o a_2 v_2 o a_3 v_3 o a_4 v_4.

If (x,$f^k(x_1)$, a) ε v_1 : (y,$g^k(x_2)$, a) ε v_2 :
(f(x), $f^k(x_3)$, a) ε v_3 : (g(y), $f^k(x_4)$ ε v_4, where each
a_1 (I = 1,2,3,4) are independent of x,y,a,z_1,$z_{2,}$ $z_{3,}$ z_4

and v_1, v_2, v_3, v_4, $a_1 \geq$ O for each i = 1,2,3,4 and

$$\sum_{i=1}^{4} a_1 < 1, k \geq 1 \text{ (k is an integer).}$$

Then f and £ have a unique common fixed point in x.

Proof: Suppose v be any member of V and p the Minkowaki's paeudc 2 – metric of v.
Put p(x, f^k (z_1), a) = r_1 : p(y, $g^k(z_2)$, a) = r_2
P(f(x), $f^k(z_3)$, a) = r_3 : p (g(y), $g^k(z_4)$, a) = r_4.

Now for any $\varepsilon > O$, we have

$(x, f^k(z_1), a) \varepsilon (r_1 + \varepsilon) v$; $(x, g^k(z_2), a) \varepsilon (r_2 + \varepsilon) v$,
$(f(x), f^k(x_3), a) \varepsilon (r_3 + \varepsilon) v$; $(g(y), g^k(z_4), a) \varepsilon (r_3 + \varepsilon) v$

Thus by given condition we have

$(f(x), g(y), a) \varepsilon a_1 (r_1 + \varepsilon) v \text{ o } a_2(r_2 + \varepsilon) v \text{ o } a_3 (r_3 + \varepsilon) v \text{ o } a_4 (r_4 + \varepsilon) v$
Then by lemma (1.1.10) and since ε is arbitrary, we have

$P(f(x), g(y), a) \leq a_1 p(x, f^k(z_1), a) , a_2 p(y, g^k(z_2), a), a_3 p (f(x), f^k(z_3), a) + a_4 p(g(y), g^k(z_4), a)$ …………………… (1)

Now for the arbitrary x,w ε X, put

$X = g^k(z), y = f^k(w), z_1 = w, z_2 = z, z_3 = g(w)$, then we have

$$P(fg^k(z), gf^k(w), a) \leq a_1 p (g^k(z), f^k(w), a) + a_2 p (f^k(w), g^k(w), a) + a_3 p(fg^k(z), f^k(w), a) + a_4 p(gf^k(w), g^k f(w), a)$$

$$\leq ap(g^k x), f^k(w), a) \text{ ……………….. (2)}$$

Where $a = \dfrac{a_1 + a_2}{1 - a_3 - a_4}$

Let x_o be arbitrary, Define a sequence $\{x_n\}$ as follows

$$X_n = \begin{cases} f(x_{n-1}) \text{ when } n \text{ is odd} \\ g(x_{n-1}) \text{ when } n \text{ is even} \end{cases}$$

In view of given condition (ii) we observe that

$X_{2n} = f^n g^n (X_o)$ and $x_{2n+1} = f^{n+1} g^n(x_o)$. Let $n > t$

Which gives $n = t + g$ for some integer $q > 1$, then proceeding as in previous theorem we have

$P(x_{2n}, x_{2n+1+,} a) \leq a^{2n-2t} p(X_{2t}, X_{2t+1}, a)$

Now for $m > n > t$ then again proceeding similar to the previous theorem we can show that $\{x_n\}$ is a Cauchy sequence. Since x is sequentially complete Hausedorff

Space, there exists u ε X such that u = $\lim_{n \to \infty} x_n$.

For any odd positive integer h, we have

P(u, g(u), a) ≤ p(u, g(u), x_h + p (u, $x_{h,}$ a) + p(x_h, g(u), a)
= p(u,g (u), x_h) + p(u,$x_{h,}$,a) p(f(x_{h-1}), g(u),a)

Taking x = x_{h-1}, y = u, $z_1 = g^k (x_{h-1})$, $z_2 = f^k (x_{h-1})$.

In (1) and using the above inequality we get

P(u, g(u), a) ≤ p(u, g(u), x_h + p (u, $x_{h,}$ a) + a_1 p(x_{h-1}, $f^k g^k(x_{h-1})$, a
+ a_2 p(u,g $g^k f^k (x_{h-1})$, a) + a_3p(f(x_{h-1}) , $f^k g^k((x_{h-1})$, a
+ a_4 p (g (u), $g^k f^k (x_{h-1})$, a)
= p (u,g(u), x_h) + p(u, x_h, a) + a_1 p(x_{h-1}, $x_{h+2k-1,}$ a)
+ a_2 p(u, $x_{h+2k-1,}$a) + a_3 p($x_{h,}$ $x_{h+2k-1,}$ a)
+ a_4 p(g(u), $x_{h+2k-1,}$ a)

Then h $\to \infty$, x_h, x_{h-1}, x_{h+2k-1} all tends to u.

Therefore p(1-a_4) p(u, g(u), a) ≤ O which implies

That p(u, g(u), a) = O. Hence (u, g(u), a) ε v.

Since v is arbitrary and X is a Hausdorff space.

Therefore, we have u = g(u). Similarly u = f(u).

Thus u is the common fixed point of f and g. For the uniquences of u. Let $\bar{u} \neq$ u be such

that

f ($\bar{u}$) = g($\bar{u}$) = ($\bar{u}$). On putting x = u = z_2 = z_4

and y = $\bar{u}$ = z_1 = z_3 in (1) the desired result follows.? //

(7.2.5) <u>THEOREM:</u> Let f and g be two operators such that

(i) f, g map x into itself
(ii) f.g = g, f

(iii) for all x,y,z_1,z_2,z_3,z_4, ε X and for every a ε X and each p ε P, any ten members v_1, v_2, v_3 …………… v_{10} in v

$(f(x), g(y),a)$ ε $a_1 v_1$ o $a_2 v_2$ o $a_3 v_2$ o $a_3 v_3$ o $a_4 v_4$ o $a_5 v_5$ o $a_6 v_6$ o $a_7 v_7$ o $a_8 v_8$ o $a_9 v_9$ o $a_{10} v_{10}$.

If $(x,f^k(z_1), a)$ ε v_1 : $(y, g^k(z_2), a)$ ε v_2 : $(f(x), f^k(z_3), a)$ ε v_3 $(g(y), g^s(z_4), a)$ ε v_4 : $(x,g^s(z_2), g(y), a)$ ε v_5 : (x,y,a) ε v_6 : $(f(x), g^s(z_4), a)$ v_7 : $(f^k(z_3), g(y), a)$ ε V_8 : $(f^k(x_3), g^s(z_4), a)$ ε v_9 $(g, f^k(z_1), a)$ ε v_{10}.

Where each a_1 (1 = 1,2……………….10) are independent of

X,y,a, z_1, z_2,z_3, z_4 and v_1's (I = 1,2 ………………10)

With each $a_1 \geq O$ (i = 1,2, ……………..10), $\sum_{i=1}^{10} a_1 < 1$ and $s,k \geq 1$.

Then f and g have a unique common fixed point in X.

<u>Proof:</u> Let v be any member of V and p is the Minkowski's pseudo – 2 – metric of v. Write
$P(x,f^k(x_1), a) = r_1$: $p(y,g^s(z_2), a) = r_2$:
$P(f(x), f^k(x_3), a) = r_3$: $p(g(y), g^s(z_4), a) = r_4$:
$P(x,g^s(z_5), a) = r_5$: $p(x,y,a) = r_6$: $p(f(x), g^s(z_4), a) = r_7$:
$P(f^k(z_3), g(y),a) = r_8$: $p(f^k(z_3), g^s(z_4), a) = r_9$:
$P(y, f^k(z_1), a) = r_{10}$.

For any arbitrary ε > o, we have $(x,f^k(z_1), a)$ ε $(r_1 + \varepsilon)$ v;

$(y,g^s(z_2), a)$ ε $(r_2 + \varepsilon)$ v ; $(f(x), f^k(r_3), a)$ ε $(r_3 + \varepsilon)$ v ;
$(g(y), g^s (z_4),a)$ ε $(r_4 + \varepsilon)$ v : $(x,g^s(z_2), a)$ ε $(r_5 + \varepsilon)$ v ;
(x,y,a) ε $(r_6 + \varepsilon)$ v ; $(f(x), g^s(z_4), a)$ ε $(r_7 + \varepsilon)$ v;
$(f^k(z_3), g(y), a)$ ε $(r_8 + \varepsilon)$ v : $(f^k(z_3), f^8(z_4), a)$ ε $(r_9 + \varepsilon)$ v:
$(y, f^k(z_1), a)$ ε $(r_{10} + \varepsilon)$ v. Thus from given condition

We have

$(f(x), g(y), a)$ ε $a_1 (r_1 + \varepsilon)$ v o $a_2 (r_2 + \varepsilon)$ v o a_3 $9r_3 + \varepsilon)$ o
$a_4((r_4 + \varepsilon)$ v o $a_5 (r_5 + \varepsilon)$ v o $a_6 (r_6 + \varepsilon)$ v o
$a_7 (r_7 + \varepsilon)$ v o $a_8 (r_8 + \varepsilon)$ v o $a_9 (r_9 + \varepsilon)$ v o
$a_{10} (r_{10} + \varepsilon)$ v.

Now by lemma (1.1.10) and since ε is arbitrary, thus we have

$$\begin{aligned} p(f(x), g(y),a) \le\ & a_1\, p(x,f^k(x_1),a) + a_2\, p(y,g^s)(x_2),a) \\ & + a_3 p(f(x), f^k(z_3), a) + a_4\, p(g(y), g^s(z_4), a) + \\ & + a_5 p(x,g^s(x_2),a) + a_6\, p(x,y,a) + \\ & + a_7\, p(f(x), g^s(z_4), a) + a_8\, p(f^k(z_3), g(y),a) \\ & + a_9\, p(f^k(z_3)\, g^s(z_4), a) + a_{10} p(y, f^k(x_1), a) \qquad \ldots\ldots\ldots\ldots (1) \end{aligned}$$

For arbitrary x,w in X, Put $x = g^s(z)$, $y = f^k(w)$,

$X_1 = w$, $z_2 = z$, $z_3 = g(w)$, $z_4 = f(z)$ in (1) we get

$$\begin{aligned} P(f(g^s(z), g(f^k(w)),a) \le\ & a_1 p(g^s(z), f^k(w), a) + \\ & + a_2\, p(f^k(w), g^s(z), a) \\ & + a_3\, p(f(g^s(z), f^k(g(w)), a) \\ & + a_4\, p(g(f^k(w)), g^s(f(z)), a) + \\ & + a_5\, p(g^s(z), g^s(z), a) + \\ & + a_6\, p(g^s(z), f^k(w), a) + \\ & + a_7\, p(f(g^s(z)), g^s(f(z)), a) + \\ & + a_8\, p(f^k(g(w)), g(f^k(w)), a) + \\ & + a_9\, p(f^k(g(w)), g^s(f(z)), a) \\ & + \quad a_{10}\, p(f^k(w), f^k(w), a) + \end{aligned}$$

$$\text{i.e. } p(f(g^s(z)), g(f^k(w)),a) \le \left(\frac{a_1 + a_2 + a_6}{1 - a_3 - a_4 - a_5}\right) p(g^s(z), f^k(w), a)$$

$$\le a\, p(g^s(z), f^k(w), a) \ \ldots\ldots\ldots\ldots (2)$$

$$\text{Where } a = \frac{a_1 + a_2 + a_6}{1 - a_3 - a_4 - a_9} < 1$$

Let x_o ε X be arbitrary. Consider the sequence (x_n) as follows:

$$X_n = \begin{cases} f(x_{n-1}) \text{ when n is odd} \\ g(x_{n-1}) \text{ when n is even} \end{cases}$$

In view of condition (ii) we observe that

$X_{2n} = f^n g^n (x_o)$ and $x_{2n-1} = f^{n+1} g^n (x_o)$

Let n > q which gives n = q + 1, for some integer I > 1

We have $p(x_{2n,} x_{2n+1}, a) \le a^{2n-2q}\, p(x_{2q}, x_{2q+1}, a)$

Now, for $m > n > q$, we can show that $\{x_n\}$ is a Cauchy sequence by similar process as done in previous theorem. Since X is sequentially complete Hausdorff space. Therefore exists a number u ε X such that $\lim_{n \to \infty} x_n = u$. For any positive integer h we have

$$P(u, g(u), a) \leq p(u,g(u), x_h) + p(u, x_h,a) + p(x_h, g(u), a)$$

$$= p(u,g(u), x_h) + p(u,x_h,a) + p(f(x_{n-1}), g(u), a)$$

Now taking $x = x_{h-1}$, $y = u$, $z_1 = g^a(x_{h-1}) = z_3$.

$X_2 = f^k(x_{h-1}) = x_4$ in (1) and using in the above

Inequality we have

$$\begin{aligned}
P(u,g(u), a) &\leq p(u,g(u), x_h) + p(u, x_h, a) + \\
&+ a_1 p(x_{h-1}, f^k(g^s(x_{h-1})), a) \\
&+ a_2\, p(u, g^s(f^k(x_{h-1})), a) + \\
&+ a_3\, p(f(x_{h-1}), f^k(g^s(x_{h-1})), a) + \\
&+ a_4\, p(g(u), g^s(f^k(x_{h-1})), a) + \\
&+ a_5\, p(x_{h-1}, g^s(f^k(x_{h-1})), a) + \\
&+ a_6\, p(x_{h-1}, u, a) + \\
&+ a_7\, p(f(x_{h-1}), g^s(f^k(x_{h-1})), a) + \\
&+ a_8\, p(f^k(g^s(x_{h-1})), g(u), a) \\
&+ a_9\, p(f^k(g^s(x_{h-1})), g^s(f^k(x_{h-1})), a) + \\
&+ a_{10}\, p(u, f^k(g_s(x_{h-1})), a) \\
&= p(u,g(u), x_h) + p(u,x_h,a) + a_1 p(x_{h-1}, x_{h+k+s-1}, a) \\
&+ a_2 p(u,x_{h+k+s-1},a) + a_3 p(x_h, x_{h+k+s-1},a) + \\
&+ a_4 p(g(u), x_{h+k+s-1}, a) + a_5 p(x_{h-1},x_{h+k+s-1},a) \\
&+ a_6 P(x_{h-1},u,a) + a_7\, p(x_h, x_{h+k+s-1}, a) + \\
&+ a_8\, p9x_{h+k+s-1}, g(u), a) + a_{10} p(u,x_{h+k+s-1},a)
\end{aligned}$$

Then $h \to \infty$, x_h, x_{h-1}, $x_{h+k+s-1}$ all tends to u.

Thus we have

$$\begin{aligned}
p(u,g(u), a) \quad \leq \quad & p(u,u,a) + p(u,u,a) + \\
& a_1\, p(u,u,a) + a_2\, p(u,u,a) +
\end{aligned}$$

a_3 p(u,u,a) + a_4 p(g (u),u,a) +
a_5 p (u, u, a) + a_6 p(u, u, a) +
a_7 p(u, u, a) + a_8 p(u, g(u), a) +
a_{10} p(u, u,a) +

i.e., $(1-a_4 - a_8)$ p(u, g(u), a) $\leq$ O which gives

p(u, g(u), a) = O as $\sum_{i=1}^{10} a_1 < 1$ and p(u,g(u),a) …… O.

Hence, (u, g(u), a) ε v.

Since v being arbitrary and X being Hausdorff space, we have u = g(u). Similarly u = f(u). Thus us is the common fixed point of f and g. For the uniqueness of u, let $u_o \neq u$, be a point such that $f(u_o) = g(u_o) = u$. On putting $x = u = x_2 = z_4$ and $y = u_o = x_1 = z_3$ in (1) we get the desired result. //

<u>Remarks;</u>

(1) If we put p = {d} without 2 – uniform space, Theorem 1.2.1 gives theorem of Lal and Singth [34].

(2) If we put p = {d} without 2 – uniform space, Theorm 1.2.2 to Theorem 1.2.5 give extended form of results of Das and Sharma [11]. Singh and Singh [55(b)] etc. in 2 – metric space.

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