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Some Applications of Fixed Point Theory

B.Nageswara Rao
Department of Basic Science and Humanities (Mathematics),
Costal Institute of Technology and Management, Narapam, Kothavalasa-535183,
Vijayanagaram district, A.P., India

Abstract

This paper deals with some new applications of some known fixed point theorems. In section one, we have obtained a result on existence of a solution of a pair of equations in Hausdorff space under some conditions. We have also discussed some applications to non-linear integral equations.

INTRODUCTION:

In section two we have discussed some applications to flow in a 2 – Banech space.

(8.1) In this section first we construct a solution of the following type of simultaneous equations defined on a compact Hausdorff space X.

(8.1.1) $T_x = S_y$ and $K_y = L_x$, where T,S, K and L are functions of X into X satisfying certain conditions.

(8.1.1) <u>THEOREM:</u> Let X be a compact Hausdorff space. Let T,S,K, L be continuous mappings of X into itself. Let F be a continuous symmetric mapping of X x X x X into the set of non-negative reals such that for all x,y,a ε X, F(x,y,a) = O If any x = y and let for all x,y,a ε X, T,S,L,K satisfy.

(i) $F(Tx, Ty\ a) > a\ F(x,y,a)$
(ii) $F(Sx, Sy, a) < \beta\ F(x,y,a)$, $x \neq y \neq a$
(iii) $F(Kx, Ky,a) > a_1\ F(x,y,a)$

(i) $F(Lx, Ly, a) < \beta\ F(x,y,a)$, $x \neq y \neq a$

Where $a > \beta > a_1 > \beta_1 > O$, $Tx = XX = X$. Then there exists a unique solution of the equation (8.1.1), i.e, $Tx^* = Sy^*$ and $Ky^* = Lx^8$.

Proof: Before giving the proof of the theorem we state a lemma :

1.2) LEMMA: Let X be a Haudorff space and let T_1 and T_2 be two continuous mapping of X into itself. Let F be a continuous symmetric mapping of X x X x X into the set of non-negative reals such that $F(x,y,z) = O$ if any of two x,y,z are equal $F(T_1^p(x), T_2^q(y), a) < \gamma_1 F(x,y,a) + \gamma_2 F(x,T_1^p(x), a) + \gamma_3 F(y, T_2^q(y), a)$

For every distinct x,y,z in X, where p> o, q>o are integers and $\gamma_1, \gamma_2, \gamma_3$ are non-negative real numbers such that $\gamma_1 + \gamma_2 + \gamma_3 < 1$. If for some x_o ε X the sequence $\{x_n\}$ consisting of distinct point x_o, where $x_{2n+1} = T_1^p x_{2n} + x_{2n+2} = T_2^q x_{2n+1}$, n = o,1,2 …… has a convergent subsequence, then T_1, T_2 have a unique common fixed point.

Proof of the lemma: it follows from [41 ( )].

Proof of the main theorem:

(i) $T_x = T_y \Rightarrow F(T_x, T_y, a) = O \Rightarrow F(x,y,a) = O \Rightarrow x = y \Rightarrow T^{-1}$ exists.

And from (iii) $Kx = Ky \Rightarrow F(Kx, Ky, a) = O \Rightarrow F(x,y,a) = O \Rightarrow x = y$

$\Rightarrow K^{-1}$ exists.

Now for $x \neq y \neq a$, $F(x,y,a) = F(TT^{-1} x, TT^{-1} y, a)$

$$\geq a F(T^{-1} x, T^{-1} y, a)$$

Therefore, $F(T^{-1} Sx, T^{-1} Sy, a) \leq \frac{1}{a} F(Sx, Sy, a)$

$$\leq \frac{\beta}{a} F(x,y,a) \text{ [from (ii)]}$$

Similarly, $F(K^{-1} L_x, K^{-1} L_y, a) \leq \frac{\beta_1}{a_1} F(x,y,a)$, $x \neq y \neq a$.

We now define a map J from X x X x X into itself by $J(x,y,a) = (T^{-1} Sy, K^{-1} 6_x ,a)$ …………. (x,y,a) ε X^3 , $x \neq y$. And let F^v be a mapping of Y x Y x Y to R_+ where Y = X x X x which is defined by

$F^1 ((p_1+ p_2 + p_3), (q_1 + q_2 q_3) (a_1+ a_2 + a_3) = F(p_1, q_1, a_1) +$

$$F(p_2, q_2, a_2) +$$

$$F(p_3, q_3, a_3) \text{ for all}$$

$(p_1, + p_2 + p_3), (q_1 + q_2 q_3) (a_1+ a_2 + a_3) \varepsilon X^3$.

It is clear that F^v is continuous and symmetric with $F^v(y_1,y_2,y_3) = O$, when at least two of $y_1,y_2,y_3 \varepsilon Y$ are.

Now for any $(x_1, y_1,a_1), (x_2, y_2,a_2), (x_3, y_3,a_3) \varepsilon X \times X \times X$ with $(x_1, y_1,a_1) \neq (x_2, y_2,a_2) \neq (x_3, y_3,a_3)$, we have

$F^v (J(x_1, y_1,a_1), J(x_2, y_2,a_2), J(x_3, y_3,a_3))$

$= F ((T^{-1} Sy_1, K^{-1} Lx_1, a_1), (T^{-1} Sy_2, K^{-1} Lx_2, a_2), (T^{-1} Sy_3, K^{-1} Lx_3, a_3))$

$= F(T^{-1} Sy_1,T^{-1} Sy_2, T^{-1} Sy_3) + F(K^{-1} Lx_1, K^{-1} Lx_2, K^{-1} Lx_3)$

$+ F(a_1, a_2, a_3)$

$$\leq \frac{\beta_1}{a_1} F(y_1,y_2,y_3) + \frac{\beta_2}{a_2} F(x_1,x_2,x_3) + \frac{\beta_3}{a_3} F(a_1,a_2,a_3)$$

$$\leq \max. \left(\frac{\beta_1}{a_1}, \frac{\beta_2}{a_2}, \frac{\beta_3}{a_3}\right) F (x_1,y_1,a) (x_2,y_2,a) (x_3,y_3,a))$$

Here $\frac{\beta_1}{a_1} < 1$, $\frac{\beta_2}{a_2} < 1$, $\frac{\beta_3}{a_3} < 1$.

Hence J satisfies all the conditions of the lemma (8.1.2) with $T_1 = T_2 = J$, $p = q = 1$.

$\gamma_1 = \max \left\{ \frac{\beta_1}{a_1}, \frac{\beta_2}{a_2}, \frac{\beta_3}{a_3} \right\}$ and $\gamma_2 = \gamma_3 = O$. Here the convergence criterion in lemma (8.1.2) is automatically satisfied since the underlying space is compact.

Therefore J has a unique fixed point and this completes the proof of the theorem. //

Now we discuss applications of fixed point theorems to non-linaar integral equations.

Let I be an interval of the real axis and let X denote the vector space of bounded, continuous and complex valued functions on I. Let X be a complete 2 – metric space induced by suitable 2 – normed space.

Let $f : I \times I \times C \to C$ be a given function. Assumed that function $a \to f(t,s,x(s))$ is integrable over I, while the function.

$1 \to \int_I f(t,s,x(s))$ ds is bounded and continuous on I, choosing any element y on X, define a map $U : X \to X$ as

$[U(x)] (t) = \int_I f(t,s,x(s))$ ds + y(s) ……………………………. (1)

A fixed point of U is the solution of x ε X of the non-linear integral equation.

$$X(t) = \int_I f(t,s,x(s))\, ds + y(t) \text{ ……………………………..(2)}$$

Under suitable conditions U or U^n will be a contraction map on X.

The existence of unique solution in X will follows from the contraction mapping theorem.

(8.1.3) EXAMPLE: Suppose f satisfies on inequality of the form /f(t,s,ξ) – f(t,s, η)] < F(t,s /ξ - η /

Then for x_1, x_2 ε X and every a ε X, we have

// $Ux_1 - Ux_2$, a // ≤ K // $X_1 - X_2$, a //, where K = sup {f_I F(t,s) ds}.

If k < 1, the operator U is a contraction map.

Similarly the existence of uniqueness of solution in X of the equation

$$X(t) = \lambda f_I f(t,s,x(s)\, ds + y(t)$$

For any given y ε X is assured provided λ is small

Enough to make k_λ < 1. //

(8.1.4) EXAMPLE: Let us consider the non-linear integral equation

$$X(t) = \lambda \int_n^t f(t,s,x(s))\, ds + y(t) \text{ …………………..(1)}$$

Where y is continuous on [a, b] and f(t,s, ξ) is continuous in the region [a,b] x [a,b] x [a,b] and satisfies the Lipschitz condition

$$\left| f(t,s,\xi) - f(t,s,\eta) \right| \le \omega \left| \xi - \eta \right|.$$

Let x = C [a,b] and U be the mapping of X into itself defined by

$$(U(x)\,(t) = \lambda \int_n^t f(t,s,x(a))\, ds + y(t),\ x \in X,\ a \le t \le b.$$

For any x_1, x_2 ε X and every a ε X, then by induction we have

$$\left|\left| (U_n x_1)(t) - (U^n x_2)(t), a \right|\right| \le \frac{1}{n} \left|\lambda\right|^n M^n \left|\left| x_1 - x_2, a \right|\right| (t-a)^n,\ a \le t \le b.$$

Or, $\left|\left|(U^n x_1) t - (U^n x_2) (t), a\right|\right| \leq \frac{1}{n} \left|\lambda\right|^n M^n (b\text{-}a)^n \left|\right| x_1 - x_2$, all

This proves that all U^n and in particular U, are continuous, and for n sufficiently.

$\frac{1}{n} \left|\lambda\right|^n M^n (b\text{-}a)^n < 1$. U^n is a contraction mapping.

For large n, Hence by contraction principle we have a unique x ε X such that $U_x = x$, i.e, x is the required unique solution of the equation91). //

(8.2) In this section we prove existence of a common fixed point of all the members of the flows called stationary points in a 2 – Banach space.

(8.2.1) DEFINITION: Let $\{x_t : t \,\varepsilon\, R^+\}$ be a set of continuous mappings of a subset Y of the 2 – Banach space X into itself, where R^+ is a commutative topological semi – group with identity element O such that $x_o(x) = x_t(x_s(x) = x_{t+s}(x)$, x ε X, s,t, ε R^+, and satisfying the continuity condition that for each t ε R^+, sup $\{\left|\left| x_t(x) - x_s(x), a \right|\right| : x, a \,\varepsilon\, Y \} \to O$ as $s \to t$. Then $\{x_t : t \,\varepsilon\, R^+\}$ is an R^+ semi-group of operators on Y and in this case $\{x_t : t \,\varepsilon\, R^+ \}$ is called a flow.

(8.2.2) THEORM: Let $\{x_t : t \,\varepsilon\, R^+\}$ be a flow on the subset Y of the 2 – Banach space X. Let $C_o \subset C_1 \subset C_2$ be convex subsets of Y such that C_o and C_2 are compact and C_1 is a neighborhood C_o relative to C_2. Suppose further that for some closed interval [a,b] with O < a < b we have

$$x_t(c_1) \subset C_2 \quad , \quad t \,\varepsilon\, [o,a]$$

$$x_t(c_1) \subset C_0 \quad , \quad t \,\varepsilon\, [a,b]$$

Then there exists a point $x_o \,\varepsilon\, C_o$ such that $x_t(x_o) = x_o$ for all t ε R^+.

Proof: For $K = \frac{2}{b-a}$, we have that for any integer k > K, there exists an integer n such that $a < \frac{m}{k} < \frac{m+k}{k} < b$. Thus $n_{m/k}(c_1) \subset C_o$ and $\frac{x_{(m+1)}}{k}(c_1) \subset C_o$. Thus by corollary $(6.1.13)_{n_1/k}$ has a fixed point $x_k \,\varepsilon\, C_o$. Since C_o is compact, there exists a limit point X_o of the set $\{x_n\}$ and a subsequence of $\{x_n\}$ which we denote by $\{y_n\}$ such that $y_n \to x_o$.

Now let n_t be any member of the flow. For any integer k > o, there exists an integer j ≥ o such that $\left| t\text{-} j/k \right|$, $1/k$. Let $g_k = H_{j/k}$. Then $g_k(x_k) = x_k$ and $g_k \to x_t$. For each $y_n = x_{mn}$ of the previously determined subsequence, let $f_n = g_{m_n}$.

Thus

$$X_t(x_o) = \lim_{n \to \infty} f_n(y_n) = \lim_{n \to \infty} y_n = x_o. //.$$

We observe that even this result is not the strongest possible one, we may also have b = a i.e. $x_n(c_1) \subset C_o$ only, and a common fixed point still exists.

(6.2.3) THEOREM: Let $\{x_t : t \varepsilon R^+\}$ be a flow on the subset f Y of the 2 – Banach space X. Let $C_o \subset C_1 \subset C_2$ be convex subsets of Y such that C_o and C_2 are compact and C_1 is a neighbourhood of C_o relative to C_2. Suppose further that for some a > o we have

$X_t(C_1) \subset C_{2=}$, t ε [o,a]

$N_a(C_1) \subset C_o$.

The there exists a point $X_o \varepsilon C_o$ such that $x_t(x_o) = x_o$ for all $t \varepsilon R^+$.

Proof: suppose C_o is compact, there exists ε > o such that $N_\varepsilon(C_o) \cap C_2 \subset C_1$. For any η < ε, η < o there exists δ > O such that $\| x_t(x) - x_a(x), b \| < n$, whenever $| t - 1 | < \delta$, for all x,b ε Y by the continuity of flow. Thus $x_t(C_1) \subset cl(N\eta(C_O) \cap C_2) = C_O$ for t ε [a - δ, a] and so there is a stationary point Z of the flow in C_O by previous theorem, Let $\{\eta_n\}$ be a null sequence. $C_O^{(n)} = N_{\eta_n}(C_O) \cap C_2$ and x_h be the stationary point of the flow in $C_O^{(n)}$ found by the above. It is clear that the sequence $\{z_n\}$ has a limit point in C_O which must be a stationary point of the flow. //

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