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Explicit and not fully implicit optimal and uniform finite difference schemes of order one for stiff initial value problems

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ABSTRACT

This paper presents explicit and not fully implicit finite difference schemes of order one for stiff initial value problems with a small parameter □ multiplying the first derivative. The schemes are modified form of classical Euler's rule of order one. And the schemes are both uniform and optimal with respect to the small parameter □ , that is, the solution of the difference scheme satisfies the error estimates of the form:

$$| u (t_i) - u_i | \leq C \min (h , \; □)$$

where C is independent of i, h and □. Here h is the mesh size and t_i is any mesh point. The explicit scheme presented in this paper solves the open problem proposed by Doolan et al., [6]. The open problem is; "Is it possible to obtain optimal or quasi-optimal methods which are not fully implicit? ". The implicit scheme presented in this paper which is not fully implicit is also a solution for the open problem. Finally numerical experiments are presented.

Keywords: initial layer , stiff initial value problems, singular perturbation problems, exponentially fitted, uniformly convergent, asymptotic expansion, finite difference schemes.
AMS (MOS) subject classification: 65F05, 65N30, 65N35, 650Y05.

1. INTRODUTION:

Consider the initial value problem on the interval $\Omega = (0, \infty)$

$$L\, u (t) \equiv □\, u' (t) + a (t)\, u (t) = f (t), \; t \in \Omega, \tag{1a}$$

$$u\, (0) = □ \tag{1b}$$

where □ > 0 is a small parameter and a and f are smooth functions on Ω . In addition, we assume that a(t) ≥ α > 0 , t Є Ω , which is sufficient to guarantee that operator L has a maximum principle and the solution u(t) of expression (1a,b) is unique and bounded.

The problem (1a,b) is a singularly perturbed equation with an initial layer at t = 0 whose width is of order □ [9, 10, 14]. We may define the corresponding reduced problem $u_0(t)$ by

$$a(t)\,u_0(t) = f(t), \quad t \geq 0 . \tag{2}$$

This is an algebraic equation which is obtained by putting □ = 0 in problem (1a).

We introduce a uniform mesh of width h on Ω with mesh points t_i= ih. We solve the problem (1a,b) by difference schemes of the form

$$L^h u_i \equiv □\, □(-\rho\, a_i^h)\, D_+ u_i + a_i^h u_i = f_i^h , \quad i \geq 0 , \tag{3a}$$

$$u_0 = □ \tag{3b}$$

where a_i^h, f_i^h and the fitting factor □ (- ρ a_i^h) are specified later. The schemes of this paper are chosen in such a way that they must solve exactly the reduced problem (2) as □ goes to zero., because the schemes which solve exactly the reduced problem (2) are expected to work well for large t. If the solution u_i of the scheme (3a,b) satisfies the reduced problem (2) exactly at the interior points, as □ goes to zero, then we call such finite difference scheme with this property as optimal.

In this paper the fitting factor will be always be chosen so that the difference scheme is uniform with respect to the small parameter, that is, if u and u_i are the solution s of (1a,b) and (3a,b) respectively, then at each node t_i, there is an error estimate of the form

$$| u(t_i) - u_i | \leq C h^p \tag{4}$$

where C and p are independent of i, h and □.

Uniformly convergent finite-difference schemes for the problem (1a, b) have been proposed by Doolan et al., [6], Carroll [2 – 5] Miller [8], Farrell (7] and Selvakumar [13]. Non-linear initial-value problems have been considered in Carroll [3], O'Reilly [11, 12] and Selvakumar [13]. The purpose of this paper is to propose two finite difference schemes for the problem (1a, b). These schemes give the answer to the open problem suggested in [6] , problem 12.3, Section 12, Part – I "Is it possible to obtain optimal or quasi-optimal methods which are not fully implicit", for the numerical solution of the problem (1a, b).

The solution of the schemes reflects the asymptotic properties of the solution of (1a, b). We derive error estimates of the form:

$$| u (t_i) - u_i | \leq C \min (h, □) \qquad (5)$$

where C Is independent of i, h and □. Schemes satisfying inequality (5) are clearly uniform of order one and optimal.

Throughout this paper ρ = h/□ and C will denote a generic constant independent of i, h and □

2. ANALYTICAL RESUTLS

In this section we collect some results concerning the solution of the problem (1a, b). The first of these show that the solution satisfies a maximum principle and hence is uniformly stable. The second lemma gives the estimates for the solution of problem (1a, b). A form of the solution of (1a, b) is also given which will be useful in the next section.

***Lemma* 2.1**
Let v (t) be a smooth function.

(a) If $v(0) \geq 0$ and $L\, v(t) \geq 0$ for $t \in \Omega$ then $v(t) \geq 0$ for all $t \in \Omega$

(b) If u(t) is the solution of the problem (1a, b) then

$$| u (t) | \leq | u(0) | + (1/\alpha) \max | f(y) |, \quad y, t \in \Omega.$$

***Proof*:** Doolan et al., Lemma 2.1 and 2.2 [6].

Lemma 2.2

Let L be the differential operator in problem (1a) and suppose that

$| v (0) | \leq C$ and

$$|(L v)^{(i)} (t) | \leq C [1 + □^{-i} \exp(- \alpha t / □)], \text{ for } 0 \leq i \leq j, \; t \geq 0.$$

Then

$$| v^{(i)} (0) | \leq C \; □^{-i}, \text{ for } 0 \leq i \leq j+1, \; t \geq 0,$$

and

$$| v^{(i)} (t) | \leq C [1 + □^{-i} \exp(- \alpha t / □)], \text{ for } 0 \leq i \leq j, \; t \geq 0.$$

Proof: Miller, Lemma 2.2 [8].

We can write [6]

$$u = v + w \qquad (6)$$

where v and w are defined by

$$v(t) = [\, u(0) - (f(0)/a(0)) \,] \exp(- a(0)\, t/□ \,) \qquad (7)$$

and

$$L\, w(t) = f(t) - L\, v(t)\,,\; w(0) = f(0)/a(0), \qquad (8)$$

To verify this, note that

$$v(0) + w(0) = u(0),\; L\,(v + w) = Lu$$

and use the uniqueness of the solution of the problem (1a).

3. VARIABLE FITTED SCHEME

In this section an explicit finite difference scheme with a variable fitting factor is proposed. The consistency, stability and convergence are discussed. The explicit finite-difference scheme for (1a, b) is

$$L^h\, u_i \equiv □\, □\,(- \rho\, a_i^{\,h}) D_+ u_i + a_i^{\,h}\, u_i = f_i^{\,h}\,, \quad i \geq 0\,, \qquad (9a)$$

$$u_0 = □ \qquad (9b)$$

where

$$a_i^{\,h} = a\,(t_i\,) \qquad (9c)$$

$$□\,(- \rho\, a_i^{\,h}) = \rho\, a_i^{\,h} \,/\, [\, 1 - \exp(- \rho\, a_i^{\,h})\,] \qquad (9d)$$

$$□\,(\rho\, a_i^{\,h}) = \exp(- \rho\, a_i^{\,h})\, □\,(- \rho\, a_i^{\,h}) \qquad (9e)$$

and

$$f_i^{\,h} = [\, a\,(t_i) \,/\, a(t_{i+1})\,]\; f\,(t_{i+1}) \qquad (9f)$$

The scheme (9a – f) is consistent with the problem (1a, b) in the sense that the discrete problem coincides with the problem (1a, b) when h approaches zero. The scheme satisfies the necessary condition for uniform convergence introduced in [6, 7], that is,

$$\text{Lim}\; □\,(- \rho\, a_i^{\,h})\,) = □\,(- \rho\, a\,(0)\,) \quad \text{as } h \to 0. \qquad (10)$$

The $a_i^{\,h}$ defined in the scheme satisfies the condition in the interval $[\, t_i\,, t_{i+1}\,]$

$$|\,(\,(1/h) \int a(t)\, dt\,) - a_i^{\,h}\, | \leq C\, h \qquad (11)$$

where C is independent of i and h. The scheme models the equation (2) exactly as □ goes to zero,

$$u_{i+1} = f(t_{i+1}) \,/\, a(t_{i+1})\,. \qquad (12)$$

That is, the scheme satisfies the necessary condition for optimal convergence, as

$\square \rightarrow 0,$

$$\lim \left(f_i^{\,h} - a_i^{\,h} \left[f(t_{i+1}) / a(t_{i+1}) \right] \right) = 0 \qquad (13)$$

exactly. And so one can expect the scheme (9a – f) to work well for large t. And the scheme is exponentially fitted, because the necessary condition (10) gives minimum requirement on the scheme to model the transient behavior of the problem (1a, b) accurately.

***Lemma* 3.1**

The finite difference operator L^h in (9a – f) have the following maximum principle : if v_i is any mesh function such that $v_0 \geq 0$ and $L^h v_i \geq 0$ for all t_i in Ω, then $v_i \geq 0$ for all $t_i \in \Omega$.

***Proof*:** Suppose v_i is such that $v_0 \geq 0$ and $L^h v_i \geq 0$ and assume that the discrete maximum principle is false. Let k be the smallest integer for which $v_k \geq 0$ and $v_{k+1} < 0$. Then

$$L^h v_k = [(\square (\rho \ a_k^{\,h})/ \rho) + a_k^{\,h}] \, v_{k+1} + a_k^{\,h} v_k < 0,$$

which is a contradiction.

***Lemma* 3.2**

The finite-difference operator L^h in (9a – f) is stable in the following sense :
if v_i is any mesh function, then

$$| v_i | \leq | v_0 | + (1/\alpha) \max | L^h v_j |, \ j \geq 0,$$

Proof: Consider the two function

$$w_i = | v_0 | + (1/\alpha) \max | L^h v_i | \pm v_i , \ j \geq 0.$$

Clearly

$$w_0 = | v_0 | + (1/\alpha) \max | L^h v_j | \pm v_0 \geq 0 ,$$

and

$$L^h w_i = a_i^{\,h} \left(| v_0 | + (1/\alpha) \max | L^h v_i | \right) \pm L^h v_i$$

$$\geq \alpha \left(| v_0 | + (1/\alpha) \max | L^h v_i | \right) \pm L^h v_i \geq 0.$$

From the discrete maximum principle for L^h we conclude that

$$w_i \geq 0$$

as required.

Uniqueness of the solution of (9a - f) follows immediately from the discrete maximum principle.

***Theorem* 3.3**

Let u and u_i be the solutions of problem (1a, b) and (9a – f) respectively. Then, at each mesh point t_i we have the following error estimate,

$$| u(t_i) - u_i | \leq C \min (h , \square)$$

where C is independent of i, h and □.

Proof: From the stability result of L^h in scheme (9a – f) it suffices to prove that

$$|\tau_i| = | L^h [u(t_i) - u_i] | \leq C \min (h , \square) ,$$

where τ_i is the truncation error of the scheme (9a–f) with respect to the problem (1a, b).

For i = 0, $\tau_0 = \square - \square = 0$.

$$\text{For } i \geq 1, \quad \tau_i = L^h [u(t_i) - u_i] = L^h u(t_i) - L^h u_i$$

$$= L^h u(t_i) - [a(t_i) / a(t_{i+1})] f (t_{i+1})$$

$$= L^h u(t_i) - [a(t_i) / a(t_{i+1})] L u (t_{i+1})$$

From (6), we can write,

$$\tau_i = [L^h v(t_i) - [a(t_i) / a(t_{i+1})] L v (t_{i+1})] + [L^h w(t_i) - [a(t_i) / a(t_{i+1})] L w (t_{i+1})]$$

$$= \tau_1 + \tau_2$$

where

$$\tau_1 = L^h v(t_i) - [a(t_i) / a(t_{i+1})] L v (t_{i+1})$$

$$= \square \; \square (\rho a_i^h) D_+ v\{t_i\} - [a(t_i) / a(t_{i+1})] \square \; v'(i_{i+1})$$

$$= \square \; \square (\rho a(t_i)) D_+ v\{t_i\} - [a(t_i) / a(t_{i+1})] \square \; v'(t_{i+1})$$

$$= \square \; \square (\rho a(t_i)) D_+ v\{t_i\} - [a(t_i) / a(t_{i+1})] \square \; \square (\rho a(0)) D_+ v\{t_i\}$$

$$= \square [\square (\rho a(t_i)) - \square (\rho a(0))] D_+ v\{t_i\} + ([a(t_{i+1}) - a(t_i)] / a(t_{i+1})) \square \; \square (\rho a(0)) D_+ v\{t_i\} \qquad (14a)$$

and

$$\tau_2 = L^h w(t_i) - [a(t_i)/a(t_{i+1})] L w(t_{i+1})$$
$$= \Box\ \Box(\rho a_i^h) D_+ w\{t_i\} - [a(t_i)/a(t_{i+1})] \Box\ w'(t_{i+1})$$
$$= \Box\ \Box(\rho a(t_i)) D_+ w\{t_i\} - [a(t_i)/a(t_{i+1})] \Box\ w'(t_{i+1}). \qquad (14b)$$

But

$$|\tau_i| = |\Box [\Box(\rho a(t_i)) - \Box(\rho a(0))] D_+ v\{t_i\}| + |([a(t_{i+1}) - a(t_i)]/a(t_{i+1}))\ \Box\ \Box(\rho a(0)) D_+ v\{t_i\}|$$
$$\le C \min(h, \Box) + C h \exp(-\rho a(0))\ v\{t_i\}$$
$$\le C \min(h, \Box) + C h \exp(-a(0)\ t_i/\Box)$$
$$\le C \min(h, \Box) \qquad (15a)$$

Since from Lemma 4.1 of [6], we have,

$$|\Box [\Box(\rho a(t_i)) - \Box(\rho a(0))] D_+ v\{t_i\}| \le C \min(h, \Box)$$

and

$$(t_i/\Box)\ \exp(-a(0)\ t_i/\Box) \le C.$$

Similarly

$$|\tau_i| \le C\ \Box\ h|w''(\theta)| + C \min(h, \Box)\ |w'(t_{i+1})| + C\ h\ \Box\ |w'(t_{i+1})| \quad \text{for some } \theta \in [t_i, t_{i+1}]$$
$$\le C\ \varepsilon \min(1, \rho) + C \min(h, \varepsilon) + C\ h\ \varepsilon$$
$$\le C \min(h, \Box) + C\ h\ \Box$$
$$\le C \min(h, \Box). \qquad (15b)$$

Since from Lemma 4.2 of [6] and from Lemma 2.2,

$$|w'(t_{i+1})| \le C, \quad h|w''(\theta)| \le C \min(1, \rho)$$

and

$$|\Box [\Box(\rho a(t_i)) - 1]| \le C \min(h, \Box).$$

From (15a, b) we have

$$|\tau_i| \le C \min(h, \Box) \quad \text{for all } i \ge 0. \qquad (16)$$

Using the stability result, we have,

$$|u(t_i) - u_i| \le C \min(h, \Box) \quad \text{for all } i \ge 0$$

Note **:** The proof of convergence adopted in the above theorem is the boot-strapping technique as in [6]. The scheme (9a– f) is optimal and uniform of order one.

4. CONSTANT FITTED SCHEME

For practical computations a constant fitting factor is of great importance because it is evaluated just once rather than at each step in the algorithm. In this section an optimal and uniform scheme with a constant fitting factor is proposed for
(1a, b). The scheme is defined as follows:

$$L^h u_i \equiv \Box\, \Box(\rho\, a(0))\, D_+ u_i + a_i^h\, u_{i+1} = f_i^h \quad i \geq 0, \tag{17a}$$

$$u_0 = \Box \tag{17b}$$

where

$$a_i^h = a(t_i), \tag{17c}$$

$$\Box(\rho\, a(0)) = \rho\, a(0) / [\exp(\rho\, a(0)) - 1], \tag{17d}$$

and

$$f_i^h = [\, a(t_i) / a(t_{i+1})\,]\, f(t_{i+1}) \tag{17e}$$

The scheme (17a – e) is consistent with the problem (1a, b). The scheme satisfies the necessary condition for uniform convergence (10) exactly. The a_i^h defined in the scheme satisfies the condition (11). The scheme models the reduced problem (2) exactly as □ goes to zero, that is, the scheme satisfies the necessary condition for optimal convergence (13) exactly. And so the scheme (17a – e) is expected to work well for large t. The discrete maximum principle for L^h and also uniform stability for the scheme (17a – e) follow from Lemmas 3.1 and 3.2 respectively. Hence the scheme (17a – e) is consistent and stable. The convergence of the scheme (17a – e) is contained in.

***Theorem* 4.1**
Let u and u_i be the solutions of problem (1a, b) and (17a – e) respectively. Then, at each mesh point t_i we have the following error estimate:

$$|u(t_i) - u_i| \leq C \min(h, \Box) \quad \text{for all } i \geq 0,$$

where C is independent of i, h and □

Proof**:** From the stability of L^h in the scheme (17a – e) it suffices to prove that

$$|\tau_i| = |L^h [u(t_i) - u_i]| \leq C \min(h, \Box),$$

where τ_i is the truncation error of the scheme (17a – e) with respect to the problem (1a, b).

For $i = 0$, $\tau_0 = \Box - \Box = 0.$

For $i \geq 1$, $\tau_i = L^h [u(t_i) - u_i] = L^h u(t_i) - L^h u_i$

$$= L^h u(t_i) - [a(t_i) / a(t_{i+1})] f(t_{i+1})$$

$$= L^h u(t_i) - [a(t_i) / a(t_{i+1})] L u(t_{i+1})$$

$$= [L^h v(t_i) - [a(t_i) / a(t_{i+1})] L v(t_{i+1})] +$$

$$[L^h w(t_i) - [a(t_i) / a(t_{i+1})] L w(t_{i+})]$$

$$= \tau_1 + \tau_2$$

where

$$\tau_1 = L^h v(t_i) - [a(t_i) / a(t_{i+1})] L v(t_{i+1})$$

$$= \Box \Box(\rho a(0)) D_+ v\{t_i) - [a(t_i) / a(t_{i+1})] \Box v'(t_{i+1})$$

$$= \Box \Box(\rho a(0)) D_+ v\{t_i) - [a(t_i) / a(t_{i+1})] \Box \Box(\rho a(0)) D_+ v\{t_i)$$

$$= ([a(t_{i+1}) - a(t_i)] / a(t_{i+1})) \Box \Box(\rho a(0)) D_+ v(t_i)$$

$$= ([(a(t_{i+1}) - a(t_i)] / a(t_{i+1})) a(0) \exp(- a(0) t_{i+1} / \Box)$$

and

$$\tau_2 = L^h w(t_i) - [a(t_i) / a(t_{i+1})] L w(t_{i+1})$$

$$= \Box \Box(\rho a(0))[D w\{t_i) - w'(t_{i+1})] + \Box [\Box(\rho a(0)) - 1] w'(t_{i+1})$$

$$+ ([(a(t_{i+1}) - a(t_i)] / a(t_{i+1})) \Box w'(t_{i+i})$$

But

$$| \tau_i | \leq C h \exp(- a(0) t_{i+1} / \Box)$$

$$\leq C h \exp(- a(0) t_i / \Box)$$

$$\leq C \min(h , \Box)$$

Since

$$(t_i / \Box) \exp(- a(0) t_i / \Box) \leq C$$

Similarly,

$$| \tau_2 | \leq C \min(h , \Box)$$

follows from the expression (15b) in Theorem 3.3. Therefore,

$| \tau_i | \leq C \min (h , □)$ for all $i \geq 0$.

Using stability result we have

$| u(t_i) - u_i | \leq C \min (h , □)$ for all $i \geq 0$.

5. NUMERICAL EXPERIMENT

This section gives numerical results for a stiff differential equation, for large values of t. All computations were performed in Pascal single precision on a Micro Vax II Computer at Bharathidasan University, Thiruchirapalli – 620 024, India. Here, define

$$\text{Absolute error} = \max | u(t_i) - u_i |, \quad i \geq 0$$

and

$$\text{Relative error} = \max | 1 - [u_i / u(t_i)] |, \quad i \geq 0$$

where $u(t_i)$ and u_i are exact and approximate solutions of (1a, b) and a difference approximation at each nodal point t_i .

The schemes [7] compared in the tables are

$$□ \; □ (- \rho \, a(t_i)) D_+ u_i + a(t_i) u_i = f(t_i) \tag{18}$$

$$□ \; □ (\rho \, a(0)) D_+ u_i + a(t_i) u_{i+1} = f(t_i) \tag{19}$$

$$□ \; □ (\rho \, a(t_{i+1})) D_+ u_i + a(t_{i+1}) u_{i+1} = f(t_{i+1}) \tag{20}$$

$$□ \; □ (\rho \, a(0)) D u_i + a(t_{i+1}) u_{i+1} = f(t_{i+1}) \tag{21}$$

$$□ [\theta \, □ (- \rho \, a(t_i)) + (1-\theta) \; □ (\rho \, a(t_i))] D_+ u_i + a(t_i) [\theta u_i + (1-\theta) u_{i+1} = [\theta f(t_i) + (1-\theta) f(t_{i+1})] \tag{22}$$

where $\theta = 0.5$.

For the test problem the schemes (9a – f) and (17a – e) are distinct. Tables 5.1 and 5.2 give absolute and relative errors for different values of h and □ for the schemes (9a – f) and (17a – e). It is observed that the schemes

(9a – f) and (17a – e) are optimal and uniform of O (min (h, □)) since for

□ = 0.00001 the absolute error is approx O (10^{-6}), whereas for □ = 0.01 the absolute error is O (h).

A comparative study is made in Tables 5.3 and 5.4. It is observed that the scheme (18) is explicit in nature, it is not optimal, but uniform. The scheme (19) is implicit in nature, it is not optimal, but uniform. The schemes (20) and (21) are implicit in nature and they are optimal and uniform. The scheme (9a – f) is explicit in nature and it is optimal and uniform. The scheme (17a – e) is implicit but not fully implicit in nature and it is optimal and uniform.

CONCLUSION

It is observed that the schemes (9a – f) and (17a – e) are better than the Backward – Euler or Trapezoidal Rule. Overall the schemes presented in this paper solves the open problem suggested in [6], Problem 12.3, Section 12, Part I. These schemes can be applied to solve systems of initial value problems. Parabolic equations can be solved using these schemes via method of lines. Boundary value problems can be solved using shooting method.

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Table 5.1
Test Problem: $u' = \lambda [(1+t)^2 - (1+t)u] + 1$, $0<t<10$, $u(0) = 1.5$, $\lambda = 1/\varepsilon$.

Scheme (9a-f)

MAXIMUM ABSOLUTE ERROR AT THE NODAL POINTS

□ \ h	1/8	1/16	1/32	1/64
0.01	8.88932E – 03	9.44424E – 03	8.94475E – 03	5.95617E – 03
0.001	8.88944E – 04	9.41277E – 04	9.69768E – 03	9.84550E – 04
0.0001	8.89301E – 05	9.41753E – 05	9.70364E – 05	9.84669E – 05
0.00001	8.94070E – 06	9.41753E – 06	9.65595E – 06	9.89437E – 06
0.000001	8.34465E – 07	9.53674E – 07	9.53674E – 07	9.53674E – 07
0.0000001	1.19209E – 07	1.19209E – 07	1.19209E – 07	1.19209E – 07

MAXIMUM RELATIVE ERROR AT THE NODAL POINTS

□ \ h	1/8	1/16	1/32	1/64
0.01	7.90167E – 03	8.88205E - 03	8.50117E – 02	5.66089E – 03
0.001	7.90119E – 04	8.89963E - 04	9.40323E – 04	9.69410E – 04
0.0001	7.90358E – 05	8.86917E – 05	9.40561E – 05	9.69172E – 05
0.00001	7.98702E – 06	8.82149E – 06	9.41753E – 06	9.77516E – 06
0.000001	7.15256E – 07	9.53674E – 07	9.53674E– 07	9.53674E – 07
0.0000001	1.19209E – 07	1.19209E – 07	1.19209E – 07	1.19209E – 07

Table 5.2
Test Problem: $u' = \lambda [(1+t)^2 - (1+t)u] + 1$, $0<t<10$, $u(0) = 1.5$, $\lambda = 1/\varepsilon$.

Scheme (17a – e)

MAXIMUM ABSOLUTE ERROR AT THE MODAL POINTS

□ \ h	1/8	1/16	1/32	1/64
0.01	8.88932E – 02	9.44424E – 03	8.94475E – 03	5.95617E – 02
0.001	8.88944E – 04	9.41277E – 04	9.69768E – 04	9.84550E – 04
0.0001	8.89301E – 05	9.41753E – 05	9.70364E – 05	9.84669E – 05
0.00001	8.94070E – 06	9.41753E – 06	9.65595E – 06	9.89437E – 06
0.000001	8.34465E – 07	9.53674E – 07	9.53674E – 07	9.53674E – 07
0.0000001	1.19209E – 07	1.19209E – 07	1.19209E – 07	1.19209E – 07

MAXIMUM RELATIVE ERROR AT THE NODAL POINTS

□ \ h	1/8	1/16	1/32	1/64
0.01	7.90167E – 02	8.88205E – 03	8.50117E – 03	5.66089E – 03
0.001	7.90119E – 04	8.85963E – 04	9.40323E – 04	9.69410E – 04
0.0001	7.90358E – 05	8.86917E – 05	9.40561E – 05	9.69172E – 05
0.00001	7.98702E – 06	8.82149E – 06	9.41753E – 06	9.77516E – 06
0.000001	7.15256E – 07	9.53674E – 07	9.53674E– 07	9.53674E – 07
0.0000001	5.96046E - 08	5.96046E - 08	5.96046E - 08	5.96046E - 08

Table 5.3
Test Problem

$u' = \lambda [(1+t)^2 - (1+t)u] + 1, 0<t<10, u(0) = 1.5, \lambda = 1/ \varepsilon$.
In Table 5.3 maximum absolute error at the nodal points are considered.

□ = 0.01

Scheme \ h	1/8	1/16	1/32	1/64
Backward Euler	3.31942E – 02	6.46458E – 02	9.74873E – 02	8.97464E – 02
Trapezoidal	1.191160E +01	1.28761E + 01	1.49662E + 01	1.99072E + 01
Scheme (18)	1.24081E – 01	6.15854E – 02	3.03383E – 02	1.47152E – 02
Scheme (19)	1.24080E – 01	6.15969E – 02	3.04699E – 02	1.50909E – 02
Scheme (20)	8.88836E – 03	9.17709E – 03	8.36967E – 03	5.72014E – 03
Scheme (21)	8.88932E – 03	9.39572E – 03	8.35967E – 03	5.72014E – 03
Scheme (22)	1.78130E – 02	1.19807E – 02	9.70137E – 03	6.21510E – 03
Scheme (17a–e)	8.88932E – 0.3	9.44424E – 03	8.94475E – 03	6.16598E – 03
Scheme (9a – f)	8.88932E – 03	9.44424E – 03	8.94475E – 03	5.95617E – 03

□ = 0.001

Scheme \ h	1/8	1/16	1/32	1/64
Backward Euler	3.53062E – 02	7.41780E - 03	1.50483E-02	2.96368E-02
Trapezoidal	1.10150E + 01	1.11662E+ 01	1.13571E+01	1.10837E+01
Scheme (18)	1.24908E – 01	6.24084E– 02	3.11584E-02	1.55344E-02
Scheme (19)	1.24908E – 01	6.24084E –02	3.11584E-02	1.55344E-02
Scheme (20)	8.88944E – 04	9.41277E –04	9.69768E-04	9.84669E-04
Scheme (21)	8.88944E – 04	9.41277E –04	9.69768E-04	9.84669E-04
Scheme (22)	8.81255E – 03	2.95317E -03	1.48833E-03	1.12200E-03
Scheme (17a–e)	8.88944E –0.4	9.41277E –04	9.69768E-04	9.84550E-04
Scheme (9a – f)	8.89444E – 04	9.41753E –04	9.69768E-04	9.84550E-04

□ = 0.00001

Scheme \ h	**1/8**	**1/16**	**1/32**	**1/64**
Backward Euler	3.55244E-05	7.53404E-05	1.55091E-04	3.14832E-04
Trapezoidal	1.10130E+01	1.10731E+01	1.10963E+01	1.17216E+01
Scheme (18)	1.24999E-01	6.24990E-02	3.12490E-02	1.56240E-02
Scheme (19)	1.24999E-01	6.24990E-02	3.12490E-02	1.56240E-02
Scheme (20)	8.94070E-06	9.41753E-06	9.65595E-06	9.89437E-06
Scheme (21)	8.94070E-06	9.41753E-06	9.65595E-06	9.89437E-06
Scheme (22)	7.82251E-03	1.96314E-03	4.98295E-04	1.32084E-04
Scheme (17a–e)	8.94070E-06	9.41753E-06	9.65595E-06	9.89437E-06
Scheme (9a – f)	8.94070E-06	9.41753E-06	9.65595E-06	9.89437E-06

Table 5.4

Test problem: $u' = \lambda [(1+t)^2 - (1+t)u] + 1$, $0<t<10$, $u(0) = 1.5$, $\lambda = 1/\varepsilon$.

In Table 5.4 maximum relative error at the nodal points are considered

□ = 0.01

Scheme \ h	**1/8**	**1/16**	**1/32**	**1/64**
Backward Euler	2.95060E-02	6.07977E-02	9.26534E-02	8.01909E-02
Trapezoidal	2.60509E+00	2.16940E+00	3.06724E+00	3.35335E+00
Scheme (18)	1.02221E-01	4.92319E-02	2.13697E-02	9.20743E-02
Scheme (19)	1.02221E-01	4.92319E-02	2.13697E-02	9.32878E-03
Scheme (20)	7.90071E-03	8.63075E-03	7.67410E-03	5.36454E-03
Scheme (21)	7.90167E-03	8.83639E-03	7.94518E-03	5.44262E-03
Scheme (22)	1.58337E-02	1.12675E-02	9.22036E-03	5.90694E-03
Scheme (17a–e)	7.90167E-03	8.88205E-03	8.50117E-03	5.86021E-03
Scheme (9a – f)	7.90167E-03	8.88205E-03	8.50117E-03	5.66089E-03

□ = 0.001

Scheme \h	**1/8**	**1/16**	**1/32**	**1/64**
Backward Euler	3.13830E-03	6.98149E-03	1.45923E-02	2.91839E-02
Trapezoidal	2.16626E+00	2.28175E+00	2.31153E+00	2.27309E+00
Scheme (18)	1.02221E-01	5.78823E-02	2.93333E-02	1.44001E-02
Scheme (19)	1.10222E-01	5.78823E-02	2.93333E-02	1.44001E-02
Scheme (20)	7.90119E-04	8.85963E-04	9.40323E-04	9.69529E-04
Scheme (21)	7.90119E-04	8.85963E-04	9.40323E-04	9.69529E-04
Scheme (22)	7.83336E-03	2.77948E-03	1.44327E-03	1.10471E-03
Scheme (17a–e)	7.90119E-04	8.85963E-04	9.40323E-04	9.69410E-04
Scheme (9a – f)	7.90119E-04	8.85963E-04	9.40323E-04	9.69410E-04

□ = 0.00001

Scheme \ h	**1/8**	**1/16**	**1/32**	**1/64**
Backward Euler	3.15905E-05	7.09295E-05	1.05442E-04	3.09944E-04
Trapezoidal	2.18499E+00	2.32822E+00	2.40933E+00	2.27309E+00
Scheme (18)	1.11102E-01	5.88141E-02	3.02933E-02	1.53748E-02
Scheme (19)	1.11102E-01	5.88141E-02	3.02933E-02	1.53748E-02
Scheme (20)	7.98702E-06	8.82149E-06	9.41753E-06	9.77516E-02
Scheme (21)	7.98702E-06	8.82149E-06	9.41753E-06	9.77516E-06
Scheme (22)	6.95336E-03	1.84762E-03	4.83155E-04	1.36057E-04
Scheme (17a–e)	7.98702E-06	8.82149E-06	9.41753E-06	9.77516E-06
Scheme (9a – f)	7.98702E-06	8.82149E-06	9.41753E-06	9.77516E-06