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On Relation between Two Absolute Index-Summability Methods

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ABSTRACT: In this paper we have established a relation between the Summability methods $X-\left|\overline{N},p_n\right|_k$ and $Y-|A|_k, k\geq 1$.

KEY WORDS: $\left|\overline{N},p_n\right|_k, X-\left|\overline{N},p_n\right|_k, X-\left|N,p_n\right|_k, X-|A|_k$ -summabilities.

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1. INTRODUCTION:

Let $\sum a_n$ be an infinite series and $\{s_n\}$ the sequence of partial sums. Let $\{p_n\}$ be a sequence of non-negative numbers with $P_n=\sum_{\nu=0}^{n} p_\nu$ for all $n\in N$. The sequence –to-sequence transformation

(1.1) $$t_n=\frac{1}{P_n}\sum_{\nu=0}^{n} p_\nu s_\nu, P_n\neq 0$$

defines $\left|\overline{N},p_n\right|$-mean of the sequence $\{s_n\}$ generated by the sequence of coefficients $\{p_n\}$. The series $\sum a_n$ is said to be summable $\left|\overline{N},p_n\right|_k$, $k\geq 1$, **[4]** if

(1.2) $$\sum_{n=1}^{\infty}\left(\frac{P_n}{p_n}\right)^{k-1}\left|t_n-t_{n-1}\right|^k<\infty \; .$$

The sequence –to-sequence transformation

(1.3) $$\tau_n = \frac{1}{P_n}\sum_{\nu=0}^{n} p_{n-\nu} s_\nu , P_n \neq 0$$

defines $|N, p_n|$-mean of the sequence $\{s_n\}$. The series $\sum a_n$ is said to be summable $|N,\ p_n|_k,\ k \geq 1$, if

(1.4) $$\sum_{n=1}^{\infty}\left(\frac{P_n}{p_n}\right)^{k-1}\left|\tau_n - \tau_{n-1}\right|^k < \infty .$$

The series $\sum a_n$ is said to be summable $X - \left|\overline{N},\ p_n\right|_k,\ k \geq 1$, if

(1.5) $$\sum_{n=1}^{\infty} X_n^{k-1}\left|t_n - t_{n-1}\right|^k < \infty$$

where $\{X_n\}$ is a sequence of positive real constants. Similarly, $\sum a_n$ is said to be summable $X - |N,\ p_n|_k,\ k \geq 1$, if

(1.6) $$\sum_{n=1}^{\infty} X_n^{k-1}\left|\tau_n - \tau_{n-1}\right|^k < \infty .$$

Let $A = (a_{nk})$ be a $\infty \times \infty$ matrix. The series $\sum a_n$ is said to be summable $X - |A|_k,\ k \geq 1$, if

(1.7) $$\sum_{n=1}^{\infty} X_n^{k-1}\left|T_n - T_{n-1}\right|^k < \infty ,$$

where the sequence –to-sequence transformation $\{T_n\}$ is given by

(1.8) $$T_n = \sum_{k=1}^{\infty} a_{nk} s_k \quad .$$

2. KNOWN THEOREMS:

Dealing with the index summability method Bor has established the following theorems:

THEOREM-A[1]:

Let $\{p_n\}$ be a sequence of positive real constants such that as $n \to \infty$

i) $np_n = O(P_n)$ ii) $P_n = O(np_n)$.

If $\sum a_n$ is summable $|C,1|_k$ then it is summable $\left|\overline{N},\ p_n\right|_k,\ k \geq 1$.

THEOREM-B[2]:

Let $\{p_n\}$ be a sequence of positive real constants such that as $n \to \infty$

i) $np_n = O(P_n)$ ii) $P_n = O(np_n)$.

If $\sum a_n$ is summable $\left|\overline{N}, p_n\right|_k$ then it is summable $|C,1|_k,\ k \geq 1$.

Subsequently Bor and Thorpe established the following result.

THEOREM-C[3]:

Let $\{p_n\}$ and $\{q_n\}$ be the sequence of positive real constants such that

i) $p_n Q_n = O(P_n q_n)$ ii) $P_n q_n = O(p_n Q_n)$.

then the series $\sum a_n$ is summable $\left|\overline{N}, q_n\right|_k$ whenever it is summable $\left|\overline{N}, p_n\right|_k$, $k \geq 1$.

Further , Tripathi established

THEOREM-D[6]:

Suppose $\{p_n\}$, $\{q_n\}$, $\{X_n\}$ and $\{Y_n\}$ are sequences of positive real constants such that

i) $q_n p_n = O(p_n q_n)$ ii) $Q_n = O(q_n X_n)$ iii) $Y_n p_n = O(p_n)$.

If $\sum a_n$ is summable $X - \left|\overline{N}, p_n\right|_k$, then it is summable $Y - \left|\overline{N}, q_n\right|_k$, $k \geq 1$.

Extending the above result , Misra, Misra and Routa established the following theorem replacing

$Y - \left|\overline{N}, q_n\right|_k$, $k \geq 1$ by $Y - \left|N, q_n\right|_k$, $k \geq 1$, in Theorem-D.

THEOREM-E[5]:

Suppose $\{p_n\}$, $\{q_n\}$, $\{X_n\}$ and $\{Y_n\}$ are sequences of positive real constants such that

i) $q_n p_n = O(p_n q_n)$ ii) $Q_n = O(q_n X_n)$ iii) $Y_n p_n = O(p_n)$

iv) $\sum_{n=\nu+1}^{n+1} \frac{1}{Q_n} = O\left(\frac{1}{Q_\gamma}\right)$ v) $\sum_{\nu=1}^{n-1} a_\nu^{\frac{k}{k-1}} = O(1)$, $k \neq 1$.

If $\sum a_n$ is summable $X - \left|\overline{N}, p_n\right|_k$, then it is summable $Y - \left|N, q_n\right|_k$, $k \geq 1$.

In what follows, in the present paper we generalize the above theorem the matrix Summability. We establish

3. MAIN THEOREM:

Suppose $\{p_n\}$, $\{q_n\}$, $\{X_n\}$ and $\{Y_n\}$ are sequences of positive real constants such that

(3.1) i) $a_{nn} P_n = O(p_n)$,

(3.2) ii) $\frac{1}{a_{nn}} = O(X_n)$,

(3.3) iii) $Y_n p_n = O(P_n)$,

(3.4) iv) $\sum_{n=r}^{m+1} \left(\frac{P_n}{p_n}\right) A_{nr} = O(a_{rr})$, where $A_{nk} = \sum_{\nu=k}^{n} a_{n\nu}$

and

(3.5) v) $\sum_{r=1}^{n} A_{nr} = O(1)$.

Then $\sum a_n$ is $Y-\left|A\right|_k$ summable whenever $\sum a_n$ is summable $X-\left|\overline{N}, p_n\right|_k$, $k \geq 1$.

4. PROOF OF THE THEOREM:

If $\{t_n\}$ is the nth $\left|\overline{N}, p_n\right|$-mean of $\sum a_n$,then

$$t_n = \frac{1}{P_n}\sum_{\nu=0}^{n} p_\nu s_\nu$$

$$= \frac{1}{P_n}\left\{p_0 a_0 + p_1(a_0+a_1)+\cdots+p_n(a_0+a_1+\cdots+a_n)\right\}$$

$$= \frac{1}{P_n}\left\{P_n a_0 + (p_n - p_0)a_1 + \cdots + (p_n - p_{n-1})a_n\right\}$$

$$= \frac{1}{P_n}\sum_{\nu=0}^{n}\left(P_n - P_{\nu-1}\right)a_\nu$$

Then

$$\Delta t_n = t_n - t_{n-1}$$

$$= \frac{1}{P_n}\sum_{\nu=0}^{n}\left(P_n - P_{\nu-1}\right)a_\nu - \frac{1}{P_{n-1}}\sum_{\nu=0}^{n-1}\left(P_{n-1} - P_{\nu-1}\right)a_\nu$$

$$= \frac{1}{P_n}\sum_{\nu=1}^{n}\left(P_n - P_{\nu-1}\right)a_\nu - \frac{1}{P_{n-1}}\sum_{\nu=0}^{n-1}\left(P_{n-1} - P_{\nu-1}\right)a_\nu$$

$$= \left(\frac{1}{P_{n-1}} - \frac{1}{P_n}\right)\sum_{\nu=1}^{n} P_{\nu-1}a_\nu$$

$$= \frac{p_{n1}}{P_n P_{n-1}}\sum_{\nu=1}^{n} P_{\nu-1}a_\nu$$

Hence,

$$\frac{P_n P_{n-1}}{p_n}\Delta t_n = \sum_{\nu=1}^{n} P_{\nu-1}a_\nu$$

$$\frac{P_{n-1}P_{n-1}}{p_{n-1}}\Delta t_{n-1} = \sum_{\nu=1}^{n-1} P_{\nu-1}a_\nu$$

Thus,

$$a_n = \frac{P_n}{p_n}\Delta t_n - \frac{P_{n-2}}{p_{n-1}}\Delta t_{n-1}$$

If $\{T_n\}$ is the nth $A=\left(a_{nk}\right)$-mean of $\sum a_n$,then

$$T_n = \sum_{k=0}^{n} a_{nk} s_k$$

$$= \sum A_{nk} a_k \ , A_{nk} = \sum_{\nu=k}^{n} a_{n\nu}$$

Then,

$$T_n - T_{n-1} = \sum_{k=0}^{n} A_{nk} a_k - \sum_{k=0}^{n-1} A_{n-1,k} a_k$$

$$= \sum_{k=1}^{n} (A_{nk} - A_{n-1,k}) a_k$$

$$= \sum_{k=1}^{n} (A_{nk} - A_{n-1,k}) \left(\frac{P_k}{p_k} \Delta t_k - \frac{P_{k-2}}{p_{k-1}} \Delta t_{k-1} \right)$$

$$= \sum_{k=1}^{n} A_{nk} \frac{P_k}{p_k} \Delta t_k - \sum_{k=1}^{n} A_{nk} \frac{P_{k-2}}{p_{k-1}} \Delta t_{k-1} - \sum_{k=1}^{n-1} A_{n-1,k} \frac{P_k}{p_k} \Delta t_k + \sum_{k=1}^{n-1} A_{n-1,k} \frac{P_{k-2}}{p_{k-1}} \Delta t_{k-1}$$

$$= S_1 + S_2 + S_3 + S_4 \text{ (say).}$$

Now,

$$\sum_{n=1}^{m+1} Y_n^{k-1} |T_n - T_{n-1}|^k \le \sum_{n=1}^{m+1} Y_n^{k-1} |S_1 + S_2 + S_3 + S_4|^k = \sum_{i=1}^{4} \sum_{n=1}^{m+1} Y_n^{k-1} |S_i|^k$$

(By Minokowski's inequality)

Our Theorem will be established if we show that $\sum_{n=1}^{m+1} Y_n^{k-1} |S_i|^k < \infty \ , \forall\, i = 1,2,3,4.$

$$\sum_{n=1}^{m+1} Y_n^{k-1} |S_1|^k = \sum_{n=1}^{m+1} Y_n^{k-1} \left| \sum_{r=1}^{n} A_{nr} \frac{P_r}{p_r} \Delta t_r \right|^k$$

$$\le \sum_{n=1}^{m+1} Y_n^{k-1} \sum_{r=1}^{n} \left(\frac{P_r}{p_r} \right)^k |\Delta t_r|^k A_{nr} \left(\sum_{r=1}^{n} A_{nr} \right)^{k-1}$$

(Using Holder's inequality)

$$= O(1) \sum_{r=1}^{m+1} \left(\frac{P_r}{p_r} \right)^k |\Delta t_r|^k \sum_{n=r}^{m+1} Y_n^{k-1} A_{nr} \text{, by (3.5)}$$

$$= O(1) \sum_{r=1}^{m+1} \left(\frac{P_r}{p_r} \right)^k |\Delta t_r|^k \sum_{n=r}^{m+1} \left(\frac{P_n}{p_n} \right)^{k-1} A_{nr} \text{, by (3.3)}$$

$$= O(1) \sum_{r=1}^{m+1} \left(\frac{1}{a_{rr}} \right)^k |\Delta t_r|^k a_{rr} \text{ , (using 3.4)}$$

$$= O(1) \sum_{r=1}^{m+1} X_r^{k-1} |\Delta t_r|^k \text{ , (using 3.2)}$$

$$= O(1) .$$

Next

$$\begin{aligned}
\sum_{n=1}^{m+1} Y_n^{k-1}|S_2|^k &= \sum_{n=1}^{m+1} Y_n^{k-1}\left|\sum_{r=1}^{n} A_{nr}\,\frac{P_{r-2}}{p_{r-1}}\,\Delta t_{r-1}\right|^k \\
&\le \sum_{n=1}^{m+1} Y_n^{k-1}\sum_{r=1}^{n}\left(\frac{P_{r-1}}{p_{r-1}}\right)^k |\Delta t_{r-1}|^k A_{nr}\left(\sum_{r=1}^{n} A_{nr}\right)^{k-1} \\
&= O(1)\sum_{r=1}^{m+1}\left(\frac{P_{r-1}}{p_{r-1}}\right)^k |\Delta t_{r-1}|^k \sum_{n=r}^{m+1} Y_n^{k-1} A_{nr}\text{ , by (3.5)} \\
&= O(1)\sum_{r=1}^{m+1}\left(\frac{P_{r-1}}{p_{r-1}}\right)^k |\Delta t_{r-1}|^k \sum_{n=r}^{m+1}\left(\frac{P_n}{p_n}\right) A_{nr}\text{ , by (3.3)} \\
&= O(1)\sum_{r=2}^{m+1}\left(\frac{P_r}{p_r}\right)^k |\Delta t_r|^k a_{rr}\text{ , (using 3.4)} \\
&= O(1)\sum_{r=2}^{m+1}\left(\frac{1}{a_{rr}}\right)^k |\Delta t_r|^k a_{rr} \\
&= O(1)\sum_{r=2}^{m+1} X_r^{k-1}|\Delta t_r|^k\text{ , (using 3.2)} \\
&= O(1)\,.
\end{aligned}$$

Also,

$$\begin{aligned}
\sum_{n=1}^{m+1} Y_n^{k-1}|S_3|^k &= \sum_{n=1}^{m+1} Y_n^{k-1}\left|\sum_{r=1}^{n=1} A_{n-1,r}\,\frac{P_r}{p_r}\,\Delta t_r\right|^k \\
&\le \sum_{n=1}^{m+1} Y_n^{k-1}\sum_{r=1}^{n-1}\left(\frac{P_r}{p_r}\right)^k |\Delta t_r|^k A_{n-1,r}\left(\sum_{r=1}^{n} A_{n-1,r}\right)^{k-1} \\
&= O(1)\sum_{r=1}^{m+1}\left(\frac{P_r}{p_r}\right)^k |\Delta t_r|^k \sum_{n=r+1}^{m+1}\left(\frac{P_n}{p_n}\right)^k A_{n-1,r}\text{ , using (3.5)} \\
&= O(1)\sum_{r=1}^{m+1}\left(\frac{1}{a_{rr}}\right)^k |\Delta t_r|^k a_{rr}\text{ , (using 3.4)} \\
&= O(1)\sum_{r=1}^{m+1} X_r^{k-1}|\Delta t_r|^k\text{ , (using 3.2)} \\
&= O(1)\,.
\end{aligned}$$

Finally,

$$\sum_{n=1}^{m+1} Y_n^{k-1}|S_4|^k = \sum_{n=1}^{m+1} Y_n^{k-1}\left|\sum_{r=1}^{n=1} A_{n-1,r}\frac{P_{r-2}}{p_{r-2}}\Delta t_{r-1}\right|^k$$

$$\leq \sum_{n=1}^{m+1} Y_n^{k-1}\sum_{r=1}^{n-1}\left(\frac{P_{r-2}}{p_{r-2}}\right)^k |\Delta t_{r-1}|^k A_{n-1,r}\left(\sum_{r=1}^{n} A_{n-1,r}\right)^{k-1}$$

$$= O(1)\sum_{n=1}^{m+1} Y_n^{k-1}\sum_{r=1}^{n-1}\left(\frac{P_{r-1}}{p_{r-1}}\right)^k |\Delta t_{r-1}|^k A_{n-1,r}$$

$$= O(1)\sum_{r=1}^{m}\left(\frac{P_{r-1}}{p_{r-1}}\right)^k |\Delta t_{r-1}|^k \sum_{n=r+1}^{m+1} A_{n-1,r} Y_n^{k-1}$$

$$= O(1)\sum_{r=1}^{m}\left(\frac{1}{a_{r-1,r-1}}\right)^k |\Delta t_{r-1}|^k a_{r-1,r-1}, \text{ (using 3.4)}$$

$$= O(1)\sum_{r=1}^{m} X_{r-1}^{k-1}|\Delta t_{r-1}|^k, \text{ (using 3.2)}$$

$$= O(1).$$

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