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# Ternary Semigroups

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**ABSTRACT**

In this paper the terms; quasi commutative ternary semigroups, normal ternary semigroups, pseudo commutative ternary semigroups are introduced. Unital element and zero element in a ternary semigroups are introduced. It is proved that every commutative ternary semigroup is a normal ternary semigroup and a Pseudo commutative ternary semigroup. It is proved that a ternary semigroup T has atmost one identity element and atmost one zero element. If A be a non-empty subset of a ternary semigroup T then it is proved that <A> = the intersection of all ternary subsemigroup of T containing A. The terms idempotent element , regular element, completely regular element, intra regular element, semisimple elemenet are introduced. It is proved that *a* be a left regular (or) lateral regular (or) right regular element of a ternary semigroup T then *a* is semisimple lelment of T. The terms arthimedean, strongly arthemedean ternary semigroups are introduced. Finally it is proved that every strongly a ternary semigroup T is an arthemedean ternary semigroup.

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**Key words** : Quasi commutative, normal, pseudo commutative, ternary subsemigroup of T generated by A, semisimple elements, regular ternary semigroup, completely regular ternary semigroup, ideal, principal ideal, idempotent, archimedean semigroup.

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## 1. INTRODUCTION:

The theory of ternary algebraic system was introduced by Lehmer [9] in 1932, but earlier such structures were studied by Kasner [7] who gave the idea of n-any algebras . Ternary semigroups are universal algebras with one associative ternary operation. Anjaneyulu.A[1], initiated the study of Pseudo commutative semigroups. Hewitt. E. and Zuckerman H.S [5] studied about Ternary opertions in semigroups. Petrch.M[11] introduced about semigroups. Rusakov.S.A [12] gave the idea of n-ary Group Theory. Santiago. M. L and Bala S.S[14] studied about ternary semigroups. Sioson.F.M [15] gave the idea of regular ternary semigroups. In this paper we initiate the study of Ternary semigroups and introduce the notions of pseudo commutative ternary quasi semigroups in ternary semigroups and charactorize the ternary semigroup.

## 2. TERNARY SEMIGROUPS :

**DEFINITION 2.1** : Let T be a non-empty set. Then T is said to be a ***Ternary semigroup*** if there exist a mapping from T×T×T to T which maps $(x_1, x_2, x_3) \rightarrow x_1x_2x_3$ satisfying the condition : $\left[\, x_1x_2x_3 \;\; x_4x_5 \right] = \left[ x_1 \;\; x_2x_3x_4 \;\; x_5 \right] = \left[ x_1x_2 \;\; x_3x_4x_5 \;\right] \; \forall \; x_i \in$ T, $1 \leq i \leq 5$.

**NOTE 2.2** : For the convenience we write $x_1x_2x_3$ instead of $x_1x_2x_3$

**NOTE 2.3** : Let T be a ternary semigroup. If A,B and C are three subsets of S , we shall denote the set ABC = $abc : a \in A, b \in B, c \in C$ .

**NOTE 2.4**: Any semigroup can be reduced to a Ternary semigroup .

**EXAMPLE 2.5**: Let T = $\left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$ Then T is a ternary semigroup under matrix multiplication.

**EXAMPLE 2.6** : Let T = $i, -i$ is a ternary semigroup under the complex ternary operation.

**EXAMPLE 2.7** : Let S = $0,1,2,3,4,5$ and abc = (a*b) *c for all a, b, c∈ S where * is defined by the table

| * | 0 | 1 | 2 | 2 | 4 | 5 |
|---|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 1 | 0 | 1 | 1 | 1 | 1 | 1 |
| 2 | 0 | 1 | 2 | 2 | 1 | 1 |
| 2 | 0 | 1 | 1 | 1 | 2 | 2 |
| 4 | 0 | 1 | 4 | 5 | 1 | 1 |
| 5 | 0 | 1 | 1 | 1 | 4 | 5 |

Then S is a ternary semigroup.

**EXAMPLE 2.8** : T = $e,a,b,c$ is a ternary semigroup under the following table

| ( ) | e | a | b | c |
|---|---|---|---|---|
| e | e | b | b | c |
| a | b | c | c | c |
| b | b | c | c | C |
| c | c | c | c | c |

$\forall$ a,b,c $\in$ T, $abc$ = ( (ab) c )

**DEFINITION 2.9** : A ternary semigroup T is said to be ***commutative*** provided for all a,b,c$\in$ T, we have abc = bca = cab = bac = cba = acb.

**EXAMPLE 2.10**: T = $0, \pm i$ with ternary operation is a commutative ternary semigroup.

**DEFINITION 2.11** : A ternary semigroup T is said to be ***quasi commutative*** provided for each *a, b, c* $\in$ T, there exists a natural number *n* such that abc = $b^n$ ac = bca = $c^n$ ba = cab = $a^n$ cb.

**THEOREM 2.12 : If T is a commutative ternary semigroup then T is a quasi commutative ternary semigroup.**

***Proof*** : Suppose that T is a commutative ternary semigroup. Let $a,b,c \in$ T.
Now $abc = b^1 ac = bca = c^1 ba = cab = a^1 cb$
T is a quasi commutative ternary semigroup.

**DEFINITION 2.13** : A ternary semigroup T is said to be ***normal*** provided abT = Tab $\forall$ $a,b \in$ T.

**THEOREM 2.14: If T is a quasi commutative ternary semigroup then T is a normal ternary semigroup.**

***Proof*** **:** It is clear from the definition of quasi commutative.

**COROLLARY 2.15 : Every commutative ternary semigroup is a normal ternary semigroup.**

***Proof*** : Let T be a commutative ternary semigroup. By theorem 2.12, T is a quasi commutative ternary semigroup. By theorem 2.14, T is a normal ternary semigroup. Therefore every commutative ternary semigroup is a normal ternary semigroup.

**DEFINITION 2.16** : A ternary semigroup T is said to be ***left pseudo commutative*** provided $abcde = bcade = cabde = bacde = cbade = acbde$ $\forall$ $a,b,c,d,e \in$ T.

**THEOREM 2.17 : If T is a commutative ternary semigroup, then T is a left pseudo commutative ternary semigroup.**

***Proof:*** Suppose that T is a commutative ternary semigroup. Then $abcde = (abc)de = (bca)de = (cab)de = (bac)de = (cba)de = (acb)de$

$\forall$ *a,b,c,d,e* $\in$ T. *abcde* = *bcade* = *cabde* = *bacde* = *cbade* = *acbde*. T is a left pseudo commutative ternary semigroup.

**NOTE 2.18** : The converse of the above theorem is not true. i.e T is a left pseudo commutative ternary semigroup then T need not be a commutative ternary semigroup.

**EXAMPLE 2.19** : Let T = *a*, *b*, *c*, *d*, . Define a ternary operation [] on T as *abc* = *a.b.c* where the binary operation '.' is defined as

| . | *a* | *b* | *c* | *d* | *e* |
|---|---|---|---|---|---|
| *a* | *a* | *a* | *a* | *a* | *a* |
| *b* | *b* | *a* | *a* | *a* | *a* |
| *c* | *a* | *a* | *a* | *a* | *a* |
| *d* | *a* | *a* | *a* | *a* | *a* |
| *e* | *a* | *b* | *c* | *d* | *e* |

It is easy to see that T is a ternary semigroup. Now T is a ternary semigroup. Now T is a left pseudo commutative ternary semigroup. But T is not a commutative ternary semigroup.

**DEFINITION 2.20**: A ternary semigroup T is said to be a ***lateral pseudo commutative*** ternary semigroup provide *abcde* = *acdbe* = *adbce* = *acbde* = *adcbe* = *abdce* for all *a,b,c,d,e* $\in$ T.

**THEOREM 2.21: If T is a commutative ternary semigroup then T is a lateral pseudo commutative ternary semigroup.**

***Proof*** : Suppose that T is a commutative ternary semigroup. Then
*abcde* = *a(bcd)e* = *a(cdb)e* = *a(dbc)e* = *a(cbd)e* = *a(dcb)e* = *a(bdc)e* for all *a,b,c,d,e* $\in$ T.
*abcde* = *acdbe* = *adbce* = *acbde* = *adcbe* = *abdce*
There fore T is a lateral pseudo commutative ternary semigroup.

**NOTE 2.22** : The converse of the above theorem is not true i.e. T is a lateral pseudo commutative ternary semigroup then T need not be a commutative ternary semigroup.

**EXAMPLE 2.23** : Consider the ternary semigroup in example 2.19. T is a lateral pseudo commutative. But T is not a commutative ternary semigroup.

**DEFINITION 2.24** : A ternary semigroup T is said to be ***right pseudo commutative*** provided *abcde* = *abdec* = *abecd* = *abdce* = *abedc* = *abced* $\forall$ *a,b,c,d,e* $\in$ T.

**THEOREM 2.25 : If T is a commutative ternary semigroup then T is a right pseudo commutative ternary semigroup.**

***Proof*** : Suppose that T is a commutative ternary semigroup. Then
*abcde* = *ab(cde)* = *ab(dec)* = *ab(ecd)* = *ab(dce)* = *ab(edc)* = *ab(ced)* for all *a,b,c,d,e* $\in$ T.
*abcde* = *abdec* = *abecd* = *abdce* = *abedc* = *abced*
T is a right pseudo commutative ternary semigroup.

**NOTE 2.26**: The converse of the above theorem is not true i.e. If T is a right pseudo commutative ternary semigroup, then T need not be a commutative ternary semigroup.

**EXAMPLE 2.27** : Consider the ternary semigroup in example 2.19. T is a right pseudo commutative. But T is not a commutative ternary semigroup.

**DEFINITION 2.28**: A ternary semigroup T is said to be ***pseudo commutative***, provided T is a left pseudo commutative, right pseudo commutative and lateral pseudo commutative ternary semigroup.

**THEOREM 2.29 : If T is a commutative ternary semigroup, then T is a pseudo commutative ternary semigroup.**

***Proof***: Suppose that T is a commutative ternary semigroup. By theorem 2.17, T is a left pseudo commutative ternary semigroup. By theorem 2.25, T is a right pseudo commutative ternary semigroup. By theorem 2.21, T is a lateral pseudo commutative ternary semigroup. T is a pseudo commutative ternary semigroup.

**NOTE 2.30** : The converse of the above theorem is not true i.e. if T is a pseudo commutative ternary semigroup , then T need not be a commutative ternary semigroup.

**EXAMPLE 2.31** : Consider the ternary semigroup in example 2.19. T is a pseudo commutative. But T is not a commutative ternary semigroup.

**DEFINITION 2.32** : An element $a$ of ternary semigroup T is said to be ***left identity*** of T provided $aat = t$ for all $t \in$ T.

**NOTE 2.33** : Left identity element $a$ of a ternary semigroup T is also called as ***left unital element.***

**DEFINITION 2.34** : An element $a$ of a ternary semigroup T is said to be a ***lateral identity*** of T provided $ata = t$ for all $t \in$ T.

**NOTE 2.35** : Lateral identity element $a$ of a ternary semigroup T is also called as ***lateral unital element.***

**DEFINITION 2.36** : An element $a$ of a ternary semigroup T is said to be a ***right identity*** of T provided $taa = t \ \forall t \in$ T.

**NOTE 2.37** : Right identity element $a$ of a ternary semigroup T is also called as ***right unital element.***

**DEFINITION 2.38** : An element $a$ of a ternary semigroup T is said to be a ***two sided identity*** of T provided $aat = taa = t \ \forall \ t \in$ T.

**NOTE 2.39** : Two-sided identity element of a ternary semigroup T is also called as ***bi-unital element***.

**DEFINITION 2.40** : An element $a$ of a ternary semigroup T is said to be an ***identity*** provided $aat = taa = ata = t \ \forall \ t \in$ T.

**NOTE 2.41:** An identity element of a ternary semigroup T is also called as ***unital element.***

**NOTE 2.42** : An element *a* of a ternary semigroup T is said to be an ***identity*** of T then *a* is left identity , lateral identity and right identity of T.

**EXAMPLE 2.43** : Let $Z_0^-$ be the set of all non-positive integers . Then with the usual ternary multiplication , $Z_0^-$ forms a ternary semigroup with identity element -1.

**THEOREM 2.44 : If *a* is a left identity , *b* is a lateral identity and *c* is a right identity of a ternary semigroup T, then *a = b = c.***

***Proof*** : Since *a* is a left identity of T, $aat = t$ for all $t \in$ T, *b* is a lateral identity of T, $btb = t$ for all $t \in$ T and *c* is a right identity of T, $tcc = t$ for all $t \in$ T $\Rightarrow aat = btb = tcc$ for all $t \in$ T and hence $a = b = c$.

**THEOREM 2.45 : Any ternary semigroup T has atmost one identity**

***Proof*** : Let *a,b,c* be three identity elements of a ternary semigroup T. Now *a* can be considered as a left identity , *b* can be considered as a lateral identity and *c* can be considered as a right identity of T. By theorem 2.44 , $a = b = c$. Then T has atmost one identity.

**NOTE 2.46** : The identity ( if exists ) of a ternary semigroup is usually denoted by 1 (or) e.

**DEFINITION 2.47** : A ternary semigroup T with identity is called a ***ternary monoid***.

**DEFINITION 2.48** : An element *a* of a ternary semigroup T is said to be a ***left zero*** of T provided $abc = a \forall b,c \in$ T

**DEFINITION 2.49** : An element *a* of a ternary semigroup T is said to be a ***lateral zero*** of T provided $bac = a \forall b,c \in$ T

**DEFINITION 2.50** : An element *a* of a ternary semigroup T is said to be a ***right zero*** of T provided $bca = a \forall b,c \in$ T

**DEFINITION 2.51**: An element *a* of a ternary semigroup T is said to be a ***two sided zero*** of T provided $abc = bca = a \forall b,c \in$ T

**NOTE 2.52** : If *a* is a two sided zero of a ternary semigroup T, then *a* is both left zero and right zero of T.

**DEFINITION 2.53**: An element *a* of a ternary semigroup T is said to be ***zero*** of T provided $abc = bac = bca = a \forall b,c \in$ T.

**NOTE 2.54** : If *a* is a zero of T, then a is a left zero, lateral zero and right zero of T.

**DEFINITION 2.55** : A ternary semigroup in which every element is a left zero is called a ***left zero ternary semigroup***.

**DEFINITION 2.56** : A ternary semigroup in which every element is a lateral zero is called a ***lateral zero ternary semigroup.***

**DEFINITION 2.57** : A ternary semigroup in which every element is a right zero is called a ***right zero ternary semigroup***.

**DEFINITION 2.58** : A ternary semigroup with 0 in which the product of any three elements equal to 0 is called ***a zero ternary semigroup*** (or) null ternary semigroup.

**EXAMPLE 2.59** : Let $0 \in$ S and $|S| > 2$. Then S with the ternary operation defined by $xyz = x$ if $x = y = z$ , 0 otherwise is a ternary semigroup with zero.

**THEOREM 2.60 : If *a* is a left zero, *b* is a lateral zero and *c* is a right zero of a ternary semigroup T , then *a* = *b* = *c*.**

***Proof*** : Since a is a left zero of T, $abc = a$ for all $a,b, c \in$ T. Since $b$ is a lateral zero of T, $abc = b$. Since $c$ is a right zero of T, $abc = c$. Therefore $abc = a = b = c$.

**THEOREM 2.61 : Any ternary semigroup element has at most one zero element.**

***Proof*** : Let $a,b,c$ be three zeros of the ternary semigroup T. Now $a$ can be considered as a left zero , $b$ can be considered as a lateral zero and $c$ can be considered as a right zero of T. By theorem 2.60 , $a = b = c$. The T has at most one zero.

**NOTE 2.62** : The zero ( if exists ) of a ternary semigroup is usually denoted by 0.

**NOTATION 2.63** : Let T be a ternary semigroup. If T has an identity, let $T^1$ = T and if T does not have an identity , let $T^1$ be the ternary semigroup T with an identity adjoined usually denoted by the symbol 1. Similarly if T has a zero, let $T^0$ = T and if T does not have a zero , let $T^0$ be the ternary semigroup T with zero adjoined usually denoted by the symbol 0.

## 3. TERNARY SUB SEMIGROUP

**DEFINITION 3.1**: Let T be ternary semigroup. A non empty subset 'S' is said to be a ***ternary subsemigroup*** of T if $abc \in$ S for all $a,b,c \in$ S.

**NOTE 3.2:** A non empty subset S of a ternary semigroup T is a ternary subsemigroup if and only if SSS $\subseteq$ S.

**EXAMPLE 3.2**: Let Z be the set of all integers. Define multiplication on Z by $[xyz]$ = min $x, y, z$ for all $x; y; z \in$ Z. Then Z is an ternary semigroup without zero. Let N be the set of all positive integers. Then N is a ternary subsemigroup of Z with a zero element 1.

**EXAMPLE 3.4** : Let T = { $a, b, c, d$ } be a semigroup under the operation **.** given by

| . | $a$ | $b$ | $c$ | $d$ |
|---|---|---|---|---|
| $a$ | $a$ | $a$ | $a$ | $a$ |
| $b$ | $a$ | $a$ | $a$ | $a$ |
| $c$ | $a$ | $a$ | $b$ | $a$ |
| $d$ | $a$ | $a$ | $b$ | $a$ |

Define the ternary operation [ ] as $[xyz] = x(yz) = (xy)z$. Then (S, [ ] ) is a ternary semigroup. Let A = {$a$ }, B = {$a$, $b$}, C = {$a$, $b$, $c$} and D = {$a$, $b$, $d$ }. Then A, B, C, D are all ternary subsemigroups of T.

**EXAMPLE 3.5** : Consider the ternary semigroup $Z^-$ under the multiplication, then $Q = Z^- \setminus \{-1\}$ is ternary subsemigroup of $Z^-$.

**THEOREM 3.6: The non-empty intersection of two ternary subsemigroups of a ternary semigroup T is a ternary subsemigroup of T.**

***Proof*** : Let $S_1, S_2$ be two ternary subsemigroups of T . Let $a,b,c \in S_1 \cap S_2$

$a,b,c \in S_1 \cap S_2 \Rightarrow$ a,b,c $\in S_1$ and $a,b,c \in S_2$

$a,b,c \in S_1$, $S_1$ is a ternary subsemigroup of T $\Rightarrow abc \in S_1$

$a,b,c \in S_2$, $S_2$ is a ternary subsemigroup of T $\Rightarrow abc \in S_2$

$abc \in S_1$, $abc \in S_2 \Rightarrow abc \in S_1 \cap S_2$. There fore $S_1 \cap S_2$ is a ternary subsemigroup of T.

**THEOREM 3.7: The non-empty intersection of any family of ternary subsemigroups of a ternary semigroup T is a ternary subsemigroup of T.**

***proof*** : Let $S_\alpha \ _{\alpha \in \Delta}$ be a family of ternary subsemigroups of T and S= $\bigcap_{\alpha \in \Delta} S_\alpha$

Let $a,b,c \in$ S. $a,b,c \in$ S $\Rightarrow a, b, c \in \bigcap_{\alpha \in \Delta} S_\alpha \Rightarrow a,b,c \in S_\alpha$ for all $\alpha \in \Delta$

$a,b,c \in S_\alpha$, $S_\alpha$ is a ternary subsemigroup of S $\Rightarrow abc \in S_\alpha$

$abc \in S_\alpha$ for all $\alpha \in \Delta \Rightarrow abc \in \bigcap_{\alpha \in \Delta} S_\alpha \Rightarrow abc \in$ S

There fore S is a ternary subsemigroup of T.

**DEFINITION 3.8**: Let T be a ternary semigroup and A be a non-empty subset of T. The smallest ternary subsemigroup of T containing A is called a ***ternary subsemigroup of T generated by A***. It is denoted by < A >.

**THEOREM 3.9: Let T be a ternary semigroup and A be a non-empty subset of T. Then <A> =** $a_1a_2....a_{n-1}a_n : n \in N, a_1, a_2....a_n \in A$

***Proof*** : Let S = $a_1a_2....a_{n-1}a_n : n \in N, a_1, a_2....a_n \in A$

Let $a,b,c \in$ S. $a \in$ S $\Rightarrow$ a = $a_1a_2a_3....a_m$ where $a_1, a_2....a_m \in A$

$b \in$ S $\Rightarrow$ b = $b_1b_2b_3....b_n$ where $b_1, b_2....b_n \in A$

$c \in$ S $\Rightarrow$ c = $c_1c_2c_3....c_r$ where $c_1, c_2....c_r \in A$

Now $abc$ = $a_1a_2a_3....a_m\, b_1b_2b_3....b_n\, c_1c_2c_3....c_r$ where $a_1, a_2....a_m, b_1, b_2....b_n, c_1, c_2....c_r \in A$ and $m,n,r \in$ N i.e. $abc \in$ S. There fore S is a ternary subsemigroup of T.

Let K be a ternary subsemigroup of T such that A $\subseteq$ K

Let $a \in$ S then $a$ = $a_1a_2a_3....a_n$ where $a_1, a_2....a_n \in A$

$a_1, a_2....a_n \in A$, A $\subseteq$ K $\Rightarrow a_1, a_2....a_n \in K$

$a_1, a_2....a_n \in K$, K is a ternary subsemigroup i.e. $a_1a_2....a_{n-1}a_n \in$ K $\Rightarrow a \in$ K. There fore $S \subseteq K$. So S is the smallest ternary subsemigroup of T containing A. Hence <A>=S.

**THEOREM 3.10 : Let T be a ternary semigroup and A be a non-empty subset of T. Then < A > = the intersection of all ternary subsemigroup of T containing A.**

***Proof*** : Let $\Delta$ be the set of all ternary subsemigroups of T containing A. Since T is a ternary subsemigroup of T containing A , T$\in \Delta$, so $\Delta \neq \emptyset$

Let $S^* = \bigcap_{s\in\Delta} S$ . Since A $\subseteq$S for all S$\in \Delta$ and A$\subseteq S^*$

By theorem 3.7 , $S^*$ is a ternary subsemigroup of T.

Since $S^* \subseteq$ S for all S$\in \Delta$ , $S^*$ is the smallest ternary subsemigroup of T containing A. There fore $S^*$ = <A>.

**DEFINITION 3.11** : Let T be a ternary semigroup. A ternary subsemigroup of T is said to be ***cyclic ternary subsemigroup*** of T if S is generated by a single element subset of T.

**DEFINITION 3.12** : A ternary semigroup T is said to be a ***cyclic ternary semigroup*** if T is cyclic ternary semigroup of T itself.

**4 . <u>SPECIAL ELEMENTS OF A TERNARY SEMIGEOUPS</u> :**

**DEFINITION 4.1 :** An element *a* of a ternary semigroup T is said to be an ***idempotent*** element provided $a^3 = a$ .

**NOTE 4.2 :** The set of all idempotent elements in a ternary semigroup T is denoted by E(T).

**DEFINITION 4.3 :** An element *a* of a ternary semigroup T is said to be a ***proper idempotent*** element provided *a* is an idempotent which is not the identity of T if identity exists.

**DEFINITION 4.4 :** A ternary semigroup T is said to be an ***idempotent ternary semigroup*** of ***ternary band*** provided every element of T is an idempotent.

**DEFINITION 4.5 :** An element *a* of a ternary semigroup T is said to be ***regular*** if there exist *x, y* ∈ T such that *axaya* = *a*.

**DEFINITION 4.6 :** A ternary semigroup T is said to be ***regular ternary semigroup*** provided every element is regular.

**THEOREM 4.7 : Every idempotent element in a ternary semigroup is regular.**

***Proof*** **:** Let *a* be an idempotent element in ternary semigroup T.
Then $a = a^2 = a.a^2 = a^2.a^2 = a.a.a.a.a$. Therefore *a* is regular element.

**DEFINITION 4.8** : A non-empty subset A of a ternary semigroup T is said to be ***left ideal*** of T if *b, c* $\in$ T , a $\in$ A implies *bca* $\in$ A.

**DEFINITION 4.9** : A non-empty subset of a ternary semigroup T is said to be a ***lateral ideal*** of T if *b, c* $\in$ T , *a* $\in$ A implies *bac* $\in$ A.

**DEFINITION 4.10** : A non-empty subset A of a ternary semigroup T is a ***right ideal*** of T if *b, c* $\in$ T , *a* $\in$ A implies *abc* $\in$ A

**DEFINITION 4.11** : A non-empty subset A of a ternary semigroup T is said to be ***ternary ideal*** or simply an ***ideal*** of T if $b, c \in$ T , $a \in$ A implies bca $\in$ A, $bac \in$ A, $abc \in$ A.

**DEFINITION 4.12** : An ideal A of a ternary semigroup T is said to be ***a principal ideal*** provided A is an ideal generated by $a$ for some $a \in$ T. It is denoted by J ($a$) (or) $\langle a \rangle$ .

**THEOREM 4.13 : A ternary semigroup T is regular then every principal ideal of S is generated by an idempotent.**

***Proof* :** Suppose T is a regular ternary semigroup. Let $< a >$ be a principal ideal of S.
Since T is regular, $\exists\ x, y \in$ T $\ni axaya = a$.
Now $(axay)^2 = axayaxayaxay = axayaxay = axay$. Let $axay = e$.
$a = axaya = axayaxaya = eea \in < e > \Rightarrow < a > \subseteq < e >$.
Now $e = axay \in < a > \Rightarrow < e > \subseteq < a >$. Therefore $< a > = < e >$ and hence every principal ideal generated by an idempotent.

**DEFINITION 4.14 :** An element $a$ of a ternary semigroup T is said to be ***left regular*** if there exist $x, y \in$ T such that $a = a^2xy$.

**DEFINITION 4.15 :** An element $a$ of a ternary semigroup T is said to be ***lateral regular*** if there exist $x, y \in$ T such that $a = xa^2y$.

**DEFINITION 4.16 :** An element $a$ of a ternary semigroup T is said to be ***right regular*** if there exist $x, y \in$ T such that $a = xya^2$.

**DEFINITION 4.17 :** An element $a$ of a ternary semigroup T is said to be ***intra regular*** if there exist $x, y \in$ T such that $a = xa^5y$.

**DEFINITION 4.18 :** An element $a$ of a ternary semigroup T is said to be ***completely regular*** if there exist $x, y \in$ T such that $axaya = a$ and $axa = xaa = aax = aya = yaa = aay = axy = yxa = xay = yax$.

**DEFINITION 4.19 :** An element $a$ of a ternary semigroup T is said to be a ***completely regular ternary semigroup*** provided every element in T is completely regular.

**THEOREM 4.20 : Let T be a ternary semigroup and $a \in$ T. Then $a$ is a completely regular element. Then $a$ is left regular, lateral regular and right regular.**

***Proof* :** Suppose that $a$ is completely regular.
Then there exist $x, y \in$ T such that $axaya = a$ and $axa = xaa = aax = aya = yaa = aay = axy = yxa = xay = yax$.
Now $a = axaya = axaay = aaaxy = a^2xy$. Therefore $a$ is left regular.
Also $a = axaya = xaaya = xaaay = xa^2y$. Therefore $a$ is lateral regular.
and $a = axaya = xaaya = xyaaa = xya^2$. Therefore $a$ is right regular.

**DEFINITION 4.21 :** An element $a$ of a ternary semigroup T is said to be a ***mid unit*** provided $xayaz = xyz$ for any $x, y, z \in$ T.

**DEFINITION 4.22 :** An element $a$ of a ternary semigroup T is said to be ***semisimple*** provided $a \in <a>^2$, that is $<a>^2 = <a>$

**DEFINITION 4.23 :** A ternary semigroup T is said to be a ***semisimple ternary semigroup*** provided every element in T is semisimple.

**THEOREM 4.24 : Let T be a ternary semigroup and $a \in T$. If *a* is completely regular then *a* is regular.**

***Proof* :** Suppose that $a$ is completely regular. Then $a = axaya$, $axa = xaa = aax = aya = yaa = aay = axy = yxa = xay = yax$ for some $x, y \in$ T. Therefore $a$ is regular.

**THEOREM 4.25 : Let T be a ternary semigroup and $a \in T$. If *a* is regular then *a* is semisimple.**

***Proof* :** Suppose that $a$ is regular. Then $a = axaya$ for some $x, y \in$ S $\Rightarrow a \in <a>^2$.
Therefore $a$ is semisimple.

**THEOREM 4.26 : Let *a* be an element of a ternary semigroup T. If *a* is left regular or lateral regular or right regular, then *a* is semisimple.**

***Proof :*** Suppose $a$ is left regular. Then $a = a^2xy$ for some $x, y \in$ S $\Rightarrow a \in < a >^2$.
Therefore $a$ is semisimple.
If $a$ is right regular, then $a = xya^2$ for some $x, y \in$ S $\Rightarrow a \in < a >^2$.
Therefore $a$ is semisimple.

**THEOREM 4.27 : Let *a* be an element of a ternary semigroup T. If *a* is intraregular then *a* is semisimple.**

***Proof :*** Suppose $a$ is intraregular. Then $a = xa^5y = xa^2a^2y$ for some $x, y \in$ S
$\Rightarrow a \in < a >^2$. Therefore $a$ is semisimple.

**DEFINITION 4.28 :** A ternary semigroup T is said to be an ***archimedian ternary semigroup*** provided for any $a, b \in$ T there exists a natural number $n$ such that $a^{2n-1} \in$ T$b$T.

**DEFINITION 4.29 :** A Semigroup S is said to be a ***strongly archimedean ternary semigroup*** provided for any $a, b \in$ T, there is a natural number $n$ such that $<a>^{2n-1} \subseteq <b>$.

**THEOREM 4.30 : Every strongly archimedean ternary semigroup is an archimedean ternary semigroup.**

***Proof* :** Suppose that T is strongly archimedean ternary semigroup. Let $a, b \in$ T. Since T is strongly archimedean ternary semigroup, there is a natural number $n$ such that $<a>^{2n-1} \subseteq <b>$. Now $a^{2n-1} \in <a>^{2n-1} \subseteq <b> \subseteq$ T$b$T. Therefore T is an archimedean ternary semigroup.

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