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Fuzzy Fixed Point Theory, Fuzzy contractive Mappings and some of its Applications

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Introduction 1.1 In this paper we present fuzzy fixed point theorems for fuzzy contractive maps. Our analysis is based on the simple observation that fuzzy fixed point results can be deduced immediately from the fixed point theory of multivalued maps with closed values. This elementary observation seems to have been overlooked in the writing. We note here also that the claim made in [1.4.9] show that their theorems are generalizations, are not justified. In Section 2, we begin by presenting a local version of Heilpem's Fuzzy fixed point theorem. This routinely leads to a simplification of Heilpem's theorem [4]; we assume a weaker contractive situation and our *a* stage sets are not tacit to be curved and compact. Also, in this section we present nonlinear alternatives of Leray – Schauder type for fuzzy contractive and fuzzy nonexpansive maps. Section 3 presents a fuzzy fixed point theory for generalized contractive maps of Kulshreshtha type. In addition, we present a homotopy result for maps of this type.

Let *(X, d)* be a metric space. By *B (O, r)* we denote the open ball in *X* centered at *x* of radius *r* and by *B(O, r)* we denote $\bigcup_{x\in C} B(x,r)$ where *O* is a subset of *X*. For *O* and *K* two nonempty closed subsets of *X*, we define the generalized Hausdorff distance *H* by

$$\text{O K}\,\varepsilon > 0; \text{O} \subseteq \text{B}(\text{K},\varepsilon),$$

$$K \subseteq \text{B}(\text{C},\varepsilon)\} \in [0,\infty],$$

A fuzzy set in X is a function with domain X and values in [0,1]. We let $\wp(X)$ denote the collection of fuzzy sets in X. Let A $\in \wp(\mathrm{X})$ and $\alpha \in [0,1]$. The a level ser of A, denoted $[A]_x$, is

$[A]_\alpha = \{\mathrm{x} : \mathrm{A(x)} \geq \alpha\}$ If $\alpha \in (0,1]$

and $A_0 = \overline{\{x : A(x) > 0\}}$.

We say that

(i) $A \in FC(X)$ if $\mathrm{A} \in \wp(\mathrm{X})$ and $[\mathrm{A}]_1$ is nonempty and closed;

(ii) $A \in FK(X)$ if $\mathrm{A} \in \wp(\mathrm{X})$ and $[\mathrm{A}]_1$ is nonempty and compact; and

(iii) $A \in FW(X)$ if $\mathrm{A} \in \wp(\mathrm{X})$ and $[\mathrm{A}]_\alpha$ is nonempty closed and bounded for each $\alpha \in [0,1]$

For A, B $\in FC(X)$, we define

D_1 (A,B) = H ($[A]_1$, $[B]_1$),

and for A, B $\in FW(X)$, we define

$D_\alpha(A,B) = H([A]_\alpha, [B]_\alpha)$ for $\alpha \in [0,1]$

and

D(A,B) = $\sup_a D_a(A,B)$.

For $A, B \in \wp(X)$, A $\subseteq$ B means that A(x) ≤B(x) for each $x \in X$. If A is a subset of X, Sits characteristic function y_A is a fuzzy set. Thus we note that a subset of X can be seen as a fuzzy set if we denote with the same symbol the subset and its characteristic function.

1.2 Fixed point theory for contractive maps [11]

We begin this section by presenting a local version of a fixed point result for contractive maps. Heilpern's fuzzy fixed point result [4] will then be generalized from our result. We also show how standard fixed point results for multi-valued contractions (in particular Nadler's fixed point theorem) could be used to deduce immediately Heilpern's fuzzy fixed point result; this elementary idea seems to have been overlooked in the writing [1, 4, 9].

1.3 Fuzzy Open set : let X be a metric space with metric d.If x_0 is a point of x and r is a positive real number, the fuzzy open sphere $F_r(x_0)$={x:d(x,x_0)<r}.

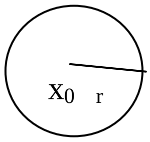

Fig: Open set

1.4 BROUWER' FIXED POINT theorem[11] : The closed unit sphere S={X:IIXII ≤} in R^n is a fixed point space.

1.5 Schauder's fixed point theorem[11]: Every convex compact subspace of a beach space is a fixed point space.

1.6 Contraction[11]: A mapping T of x into itself is called a contraction if there exist a positive real number r <1 with the property that d(Tx Ty) $\leq$r d(x y) for all points x and y in X.Where T is a maping on afuzzy set X.

1.7 THEOREM: If T is a contraction defined on a complete metric space X, then T has a unique fixed point.

Proof: let x_0 be an arbitrary point in X

And write $x_{1= T}x_0$, $x_{2=T}{}^2$ $x_{0= T}x_{1,}$ and ,

In general, $x_{n=T}{}^n$ $x_{0=T}$ x_{n-1}

if m<n , then $(x_m ,x_{n)}$= d d(T^m x_0 T^n$x_{0)}$= d(T^m $x_{0,}$ T^m T^{n-m} $x_{0)}$

$\leq r^m d(x_0\, x_{1)+}$ $d(x_1\, x_{2)+}$ $d(x_2\, x_{3)+}$ ……….+d

$d(x_{n-m}\_1, x_{n\_m)]\leq}$ $r^m d(x_0\, x_{1)[1+r+r^2}$ + …………..+ $r^n - m\_1]$

$\leq r^m \frac{d(x0\ x1)}{1\_r}$.

Since, it is evident from that { x_n }is a Cauchy sequence ,

and by the completeness of x,

there exists a point x in x such that $x_n \rightarrow$x,

We conclude that proof by showing that x is the only fixed point.

If y is also fixed point, that is Ty=y, then d(x y)=d(T_X T_y)$\leq$r d(x y);

and since r<1,this implies that d(x y)=0 or y=x.

Hence proved.

Theorem 1.8 : Let (X,d) be a complete metric space, $x_0 \in X$ and T: $\overline{B(x_0,r)} \to FC(X)$ (here r > 0). Suppose there exists a constant k $\in (0,1)$ with

$D_1(T_x,T_y) \leq kd(x,y)$ for all x, y $\in \overline{B(x_0 r}$ and

dist $(x_0,[Tx_0]_1) < (1-k)r$

holding. Then T has a fuzzy fixed point. That is there exists $x \in \overline{B(x_0,r)}$ with {x} $\subseteq Tx$ (i.e. Tx(x) =1).

Proof. Choose $x_1 \in X$ such that

$\{x\}. \subseteq Tx_0$ and $d(x_0,x_1) < (1-k)r$;

this is possible since

$[Tx_0]_1 \neq \phi$ and dist $(x_0,[Tx_0]_1) < (1-k)r$.

Now choose $\varepsilon > 0$ such that

$kd(x_0,x_1) + \varepsilon < k(1-k)r$.

Then choose $x_2 \in X$ such that $\{x\} \subseteq Tx_1$ and $d(x_1,x_2) \leq D_1(Tx_1,Tx_0) + \varepsilon$.

As a result we have $d(x_1x_2) \leq kd(x_1,x_0) + \varepsilon < k(1-k)r$ and note that $x_2 \in \overline{B(x_0 r)}$ since

$$d(x_0,x_2) \leq (1-k)r[1+k]$$
$$\leq (1-k)r[1+k+k^2+...] = r.$$

Continue this process to obtain $\{x_n\} \subseteq Tx_{n-1}$ with $d(x_n,x_{n-1}) < k^{n-1}(1-k)r$, for n = 3, 4, …. . Notice that (x_n) is a Cauchy sequence and, since X is complete, there exists $x \in \overline{B(x\ r)}$ with $\lim_{n\to\infty} x_n = x$. It remains to show $\{x\} \subseteq T_x$. Note

$$\text{dis } (x,[Tx]_1) \leq d(x,x_n) + dist(x_n,[Tx]_1)$$
$$\leq d(x,x_n) + D_1(Tx_{n-1},Tx)$$
$$\leq d(x,x_n) + kd(x_{n-1},x)$$
$$\to 0 \ as \ n \to \infty.$$

Thus $x \in \overline{[Tx]_1} = [Tx]_1$ and so $\{x\} \subseteq Tx$. □

Next we present a generalization of Heilpern's fuzzy fixed point theorem. Notice here we assume a weaker contractive condition and our α level sets are not assumed to be convex and compact.

Theorem 1.9: Let (x,d) be a complete metric space, T: X → FC (X) and suppose there exists a constant $k \in (0,1)$ with

$$D_1(Tx, Ty) \le \text{kd(x, y) for all x, y} \in \text{X}.$$

Then T has a fuzzy fixed point.

Proof. Fix $x_0 \in X$. Choose r > o so that dist $(x_0, [Tx_0]_1) < (1\text{-}k)$r. Now Theorem 2.1 guarantees that there exists $x \in \overline{B(x_0, r)}$ with {x} $\subseteq Tx$. □

Once one realizes the above elementary idea, then many results from the literature on contractive or indeed non-expansive multi-functions have fuzzy analogues. For completeness, we present fuzzy analogues of the nonlinear alternative of Leray-Schauder type for contractive and non-expansive multi-functions. We note here that we could present a more general version of Theorem 2.3 if we used the notion of contractive homotopy [2] (see Section 3 for a generalization). First recall the following two results from [2,3].

Theorem 2.0 : Let E be a Banach space, U an open subset of E, 0 $\in$U and F: $\overline{\cup} \to \text{C(E)}\alpha\ \text{k-contraction}$ (here $k \in (0,1)$) such that F (U) is bounded. Then either

(A1) F has a fixed point, or

(A2) there exists $x \in \partial\,\text{U and } \lambda \in (0,1) \text{with x} \in \lambda\,\text{Fx}$.

Theorem 2.1 : Let E = (E || . ||) be a uniformly convex Banach space and U a bounded, convex, open subset of E with $0 \in U$. Suppose F: $\overline{\cup} \to K(\text{E})$ is non-expansive with F (U) bounded (here K(E) denotes the family of nonempty, compact subsets of E and F being non expansive means $H(Fx, Fy) \le \|\text{x - y}\|$ for all x, y $\in \overline{\text{U}}$).

Then either

(A1) F has a fixed point, or

(A2) there exists $x \in \partial\,\text{U and } \lambda \in (0,1) \text{with x} \in \lambda\,\text{Fx}$.

Theorem 2.2 : Let E = (E || . ||) be a Banach space, U an open subset of E, $0 \in U$. and T: $\overline{\cup} \to FC(\text{E})$. Suppose there exists $k \in (0,1)$ with

$$D_1(Tx,Ty) \leq k \| x - y \| \text{ for all } x, y \in \overline{U} \tag{2.1}$$

and

$$T_x\left(\frac{x}{\lambda}\right) \neq 1 \text{ for all } x \in \partial U \text{ and } \lambda \in (0,1) \tag{2.2}$$

holding, and assume $[T\overline{U}]_1$ is bounded. Then T has a fuzzy fixed point, that is there exists $x \in \overline{U}$ with $\{x\} \subseteq Tx$.

Proof. Let F: $\overline{U} \to C(E)$ be given by Fx = $[Tx]_1$. We will apply Theorem 2.3. Suppose (A2) holds. Then there exists $x \in \partial U$ and $\lambda \in (0,1)$ with $x \in \lambda Fx. = \lambda[Tx]_1$ (i.e. $Tx(x/\lambda) = 1$). This contradicts (2.2) so (A1) must hold. That is there exists $x \in \partial \overline{U}$ with $x \in Fx. = [Tx]_1$. □

Theorem 2.4 and a similar argument ieldsy the following result.

Theorem 2.3: *Let E = (E,* || . || be a uniformly convex Banach space and *U a* bounded, convex, open subset of E with *0* $\in U$ and *T:* $\overline{\cup} \to FK(E)$. Suppose $D_1(Tx,Ty) \leq \| x - y \| \text{ for all } x, y \in \overline{U}$ and $Tx\left(\frac{x}{\lambda}\right) \neq 1 \text{ for all } x \in \partial U \text{ and } \lambda \in (0,1)$ hold, and assume $[T,U]_1$ is bounded. Then there exists $x \in \partial \overline{U}$ with $\{x\} \subseteq TX$.

Remark 2.4: In the above spirit it is easily seen that the nonresponsive results in [1,10] can be generalized.

Next we present some fixed point theory for maps T: FW(X) → FW (X). As we shall see a stronger contractive condition will be needed to guarantee the existence of a fixed point.

Lemma 2.5: Let *(X,d)* be a complete metric space. Then *(FW(X), D)* is a complete metric space.

Proof. Let $\{u_n\}$ be a Cauchy sequence in FW(X). Then $[u_n]_a$, for every *x*, is a Cauchy sequence in *CB(X)* (here *CB(X)* denotes the family of nonempty, closed, bounded subsets of *X)*. Recalling that *(CB(X), H)* is complete, see for example [5, p. 24], then for every *x* there exists $A_x \in CB(X)$ with

$$[u_n]_a \to A_\alpha = \bigcap_n \overline{\bigcup_{m \geq n} [u_m]_\alpha}$$

Define

$$u(x) = \sup_{x \in A_\alpha} \alpha.$$

Essentially the same analysis as in [8, p. 420-421], shows that *[u]*$_a$ = $_{Aa}$.

Theorem 2.6 : Let *(X,d)* be a complete metric space and T: FW(X) → FW(X). Suppose there exists

$$D(Tx, Ty) \le \mathrm{kD(x, y)} \quad \text{for all x, y} \in \mathrm{FW\,(X)}$$ asset Then there exists

$x \in FW(X)$ with $\mathrm{x = Tx}$.

Proof. This is immediate from Lemma 2.7 and the Banach contraction principle.

We now present a result where the map is not defined on all of *FW(X)*. It is an immediate consequence of Lemma 2.7 and a result on contractive homotopy [3].

Theorem 2.7 : Let *(X,d)* be a complete metric space. Suppose *U* is an open subset of *FW(X)* and *N* : *U* x [0, 1] → FW (X) is such that

(a) there exists $k \in (\mathrm{o},1)$ with

$$D(N(x,t), N(y,t)) \le kD(x, y)$$

for all x, y $\in \overline{\mathrm{U}}$ and $\mathrm{t} \in [0,1]$;

(b) there exists $\phi : [0,1] \to R\ with$

$$D(N(x,t), N(x,s)) \le |\,\phi(t) - \phi(x)\,|$$

for all t, s $\in [0,1]$ and $\mathrm{x} \in \overline{\mathrm{U}};$

(c) $x \ne \mathrm{N(x,t)}$ for all $\mathrm{x} \in \partial\mathrm{U}$ and $\mathrm{t} \in [0,1]$

Then N(.,0) has affixed point if and only if N (.,1) has a fixed point.

2.8 Fixed point theory for maps of Kulshrestha type

In this section we begin by presenting fixed point results for Kulshreshtha contractive maps with closed values defined on complete metric spaces. From these results we deduce right away some fuzzy fixed point theorems which simplify results of Singh and Talwar [9].

Theorem 2.9 : Let *(X,d)* be a complete metric space, $x_0 \in X$, $\mathrm{r} > 0$ and $\mathrm{F} : \overline{\mathrm{B}(\mathrm{x}_0 r)}$ → C(X). Suppose there exists $q \in (0,1)$ such that for $\mathrm{x, y} \in \overline{\mathrm{B}(\mathrm{x}_0, r)}$ we have *H(Fx, Fy)*

$$\le q \max\left\{\mathrm{d(x, y), dist\,(x, Fx), dist\,(y, Fy)}, \frac{1}{2}\left[dist(x, Fy) + dist(y, Fx)\right]\right\}$$

and

dist $\left(x_0, Fx_0\right) < (1-q)r.$

Then F has a fixed point (i.e. there exists $x, \in \overline{B(x_0, r)}$ with $x \in Fx$)

Proof. Choose $x_1 \in Fx_0$ with $d\left(x_1, x_0\right) < (1-q)r,$ so $x_1 \in \overline{B(x_0, r)}.$ Now choose $\varepsilon > 0$ such that

$$qd\left(x_1, x_0\right) + \frac{\varepsilon}{1-q} < q\,(1-q\,)\,r. \qquad (3.1)$$

Then choose $x_2 \in Fx_1$ with

$$d(x_1, x_2)$$
$$\leq H\,(Fx_0, Fx_1) + \varepsilon$$
$$\leq q \max \{d(x_0, x_1), dist(x_0, Fx_0), dist(x_1, Fx_1\},$$
$$\frac{1}{2}\left[dist(x_0, Fx_1) + dist(x_1, Fx_0)\right] + \varepsilon$$
$$\leq qd\,(x_0, x_1) + \frac{\varepsilon}{2-q};$$

this is immediate since if say the maximum of the right-hand side of the above displayed equation is $\frac{1}{2}\left[dist(x_0, Fx_1) + dist(x_1, Fx_0)\right]$, then

$$d(x_1, x_2) \leq (q/2)d(x_0, x_1) + \varepsilon$$

and so $d(x_1, x_2) \leq \frac{q}{2-q} d(x_0, x_1) + \varepsilon\left(\frac{2}{2-q}\right)$

(note *q/(2-q) <q and 2/(2-q)<1/(1-q)).* The other cases are easier. Now with ε chosen as in (3.1) we have $d(x_1, x_2) < q(1-q)r.$

Notice $x_2 \in \overline{B\left(x_0, r\right)}$ since $d(x_0, x_2) \leq (1-q)r + q(1-q)r$

$$\leq (1-q)r\left[1 + q + q^2 + \ldots\right] = r.$$

Next choose $\delta > 0$ such that

$$qd(x_1, x_2) + \frac{\delta}{1-q} < q^2(1-q)r.$$

Then choose $x_3 \in Fx_2$ with

$$d(x_2, x_3) \leq H(Fx_1, Fx_2) + \delta \leq qd\,(x_1, x_2) + \frac{\delta}{1-q}$$

$< q^2(1-q)r.$

Notice as well that $x_3 \in \overline{B(x_0, r)}$ proceed inductively to obtain $x_n \in Fx_{n-1}$, n=4, 5, … with $d(x_{n+1,}x_n) < q^n(1-q)r$ and $x_n \in \overline{B(x_0, r)}$. Now since $q \in (0,1)$ we have that (x_n) is Cauchy and so there exists $x \in \overline{B(x_0, r)}$ with $\lim_n \to \infty x_n = x$. It remains to show $x \in$ Fx. Notice

$$\text{dist } (x, Fx) \le d(x, x_n) + dist(x_n, Fx)$$

$$\le d(x, x_n) + q \max \{d(x, x_{n-1}),$$

$$\text{dist } (x, Fx) dist(x_{n-1} Fx_{n-1}),$$

$$\frac{1}{2} [\text{dist}(x, Fx_{n-1}) + dist(x_{n-1}, Fx)] \}$$

$$\le d(x, x_n) + q \max \{d(x, x_{n-1}),$$

$$\text{dist } (x, Fx) d\ (x_{n-1}, x_n),$$

$$\frac{1}{2}[d(x, x_n) + d(x_{n-1} x) + dist(x, Fx)] \}$$

Letting $n \to \infty$ gives

$$\text{dist } (x, Fx) \le d \text{ dist } (x, x_n)\ i.e. + dist(x, Fx) = 0.$$

Thus $x \in \overline{Fx} = Fx$. □

We next note that we obtain Kulshrestha's [6] fixed point result as a Corollary of Theorem 3.1, see also [7].

Theorem 3.0: *Let (X,d)* be a complete metric space and F: X → CX). Suppose there exists $q \in (0,1)$ such that for x, y $\in$ X we have

H (Fx, Fy) ≤ q max {d(x,y), dist (x, Fx), dist (y, Fy).

$$\frac{1}{2}[dist(x, Fy) + dist(y, Fx)]\}.$$

Then F has a fixed point.

Proof. Fix $x_0 \in X$. Choose $r > 0$ so that dist $(x_0, Fx) < (1-q)r$.

$\frac{Now}{B(x_0, r)}$ Theorem 3.1 guarantees that there exists $x \in$ with $x \in$ Fx. □

Next we extend the homotopy results in [2,3] for generalized contractive homotopy of Kulshrestha type.

Theorem 3.1 : *Let (X,d)* be a complete metric space and *U* open in *X*. Suppose *N*: *U* x [0, 1] $\rightarrow$(X) is a closedmap (i.e. has closedgraph) with the following satsified:

(a) $x \notin \mathrm{N}(\mathrm{x}, \mathrm{t})$ for $\mathrm{x} \in \partial \mathrm{U}$ and $\mathrm{t} \in [0,1]$;

(b) there exists $q \in (0,1)$ such that for all $\mathrm{t} \in [0,1]$ and x, $\mathrm{y} \in \mathrm{U}$ we have

$$H(N)x,t), N(y,t))$$
$$\leq \mathrm{q} \max\{\mathrm{d}(\mathrm{x},\mathrm{y}), \mathrm{dist}(\mathrm{x}, \mathrm{N}(\mathrm{x},\mathrm{t})), \mathrm{dist}(\mathrm{y}, \mathrm{N}(\mathrm{y},\mathrm{t})),$$
$$\frac{1}{2}[dist(x, N(y,t)) + dist(y, N(x,t))]\};$$

(c) there exists a continuous increasing function $\phi : [0,1] \rightarrow R$ such that

$$H(N(x,t), N(x,s)) \leq |\phi(\mathrm{t}) - \phi(\mathrm{s})| \text{ for all t, s } \in [0,1] \text{ and } \mathrm{x} \in \mathrm{U}.$$

Then N (.,0) has a fixed point if and only if N(.,1) has a fixed point.

Proof. Suppose *N(.,0)* has a fixed point. Consider $Q = \{(\mathrm{t},\mathrm{t}) \in [0,1] \mathrm{x}\, \mathrm{U} : \mathrm{x} \in \mathrm{N}(\mathrm{x},\mathrm{t})\}$.

Now *Q* is nonmpty since *N (.,0)* has a fixed point. On *Q* define the partial order

$$(t,x) \leq (\mathrm{s},\mathrm{y}) \text{ iff } \mathrm{t} \leq \mathrm{s}$$

and $d(x,y) \leq \dfrac{2[\phi(\mathrm{s}) - \phi(\mathrm{t})]}{1-q}$.

Let *P* be a totally ordered subset of *Q* and let

$t^* = \sup\{\mathrm{t} : (\mathrm{t},\mathrm{x}) \in \mathrm{P}\}$.

Take a sequence $\{(t_n, x_n)\}$ in P such that $(\mathrm{t}_n, x_n) \leq (\mathrm{t}_{n+1,} x_{n+1})$ and $\mathrm{t}_n \rightarrow t^*$. We have

$$d(x_m, x_a) \leq \frac{2}{1-q}[\phi(t_n)] \quad \text{for all } \mathrm{m} > \mathrm{n},$$

So (x_m) is a Cauchy sequence, which converges to some $x^* \in U$.*Now since N* is a closed map we have $x^* \in N(x^*, t^*)$ and also (a) implies $x^* \in \mathrm{U}$. Thus $(t^*, x^*) \in Q$. It is also in need of attention from the definition of t^* and the fact that *P* is entirely ordered that

$$(t,x) \leq (\mathrm{t}^*, x^*) \quad \text{for every } (\mathrm{t},\mathrm{x}) \in \mathrm{P}.$$

Thus (t^*, x^*) is an upper bound of *P*. By Zorn's lemma *Q* admits a maximal element $(t_0, x_0) \in$ Q.

We claim $t_0 = 1$)if our claim is true then we are finished). Suppose our claim is false. Then, choose *r > 0* and $t \in (t_0, 1)$ with

$$\overline{B(x_0, r)} \subseteq \text{U and r} = \frac{2[\phi(\text{t})] - \phi(t_0)]}{1-q} \ .$$

Notice

$$\text{dist } (x_0, N(x_0, t))$$

$$\leq \text{dist } (\text{x}_0, N(x_0, t_0)) + H(N(x_0, t_0), N(x_0, t))$$

$$\leq \phi(\text{t}) \text{ - } \phi(\text{t}_0) = \left(\frac{1-q}{2}\right) r < (1\text{-q})\text{r}.$$

Now Theorem 3.1 guarantees that *N(.,t)* has a fixed point $x \in \overline{B(x_0, r)}$. Thus $(x, t) \in \text{Q}$ and notice since $d(x_0, x) \leq r = \dfrac{2[\phi(t) - \phi(t_0)]}{1-y}$ and $t_0 < t$, we have $(t_0, x_0) < (\text{t}, \text{x})$. This contradicts the maximalist of (t_0, x_0). □

We now establish the fuzzy analogue of Theorems 3.1, 3.2 and 3.3.

Theorem 3.1: *Let (X,d)* be a complete metric space, $x_0 \in X, \text{r} > 0$ and T: $\overline{B(x_0, r)} \to \text{FC(X)}$. Suppose there exists $\text{q} \in (\text{o},1)$ such that for $\text{x, y} \in \overline{\text{B}(\text{x}_0, r)}$ we have

$$D_1(T_x, Ty)$$

$$\leq \text{q max } \{\text{d(x, y)}, \text{dist(x,[Tx]}_1), \text{dist (y,[Ty]}_1),$$

$$\frac{1}{2}[dist(x,[Ty]_1) + dist(y,[Ty]_1)] \ \}.$$

Then *F* has a fuzzy fixed point.

As an immediate consequence of Theorem 3.3 we have the following fuzzy result.

Theorem 3.2 : Let *(X,d)* be a complete metric space and *U* open *X*. Suppose *T: Q [0,1]* → FC(X) is a closed map with the following satisfied:

(a) $x \notin [\text{T(x, t)}]_1$ for $\text{x} \in \partial\text{U}$ and $\text{t} \in [9,1]$;

(b) there exists $q \in (0,1)$ such that for all $\text{t} \in [0,1]$ and $\text{x, y} \in \text{U}$ we have

$$D_1(T(x,t), T(y,t)).$$

$$\leq \text{q max } \{\text{d(x, y)}, \text{dist(x,[T(x, t)]}_1),$$

$$\frac{1}{2}[dist(x,[T(y,t)]_1)$$

$$+ dist(y,[T(x,t]_1)] \ \};.$$

(c) there exists a continuous increasing function $\phi:[0,1]\to R$ such that

$D_1(T(x,t),T(x,s)) \le |\phi(t)-\phi(s)|$.

for all $t,s \in [0,1]$ and $x \in U$.

Then *T (.,0)* has a fuzzy fixed point if and only if *T (.,1)* has a fuzzy fixed point.

FIXED POINT **3.3**

In mathematics fixed point (some times shortened to fix point, also known as an invarientpoint) of a function is a point [1] that is maped to itself by the function .A set of fixed points is sometimes called a fixed set .That is to say,c is a fixed point of the function f(x)if and only if f(c)=c.

example: f(x)=x2-3x+4 ,then 2 is a fixed point of f,because f(2)=2.

Remark: Not all functions have fixed points :for example ,if f is function defined on the real numbers as f(x)=x+1,then it has no fixed points ,since x is never equal to x+1 for any real number . In graphical terms ,a fixed point means the point (x ,f(x))is one of the line y=x, or any other words the graph of f has a point in common with the line are a pair of parallel lines.

Periodic points: the points which comeback to the same value after a finite number of iterations of the functions are known as a periodic points. affixed point is a periodi point with perod is equeal to one .in the projective geometry ,a fixed point of acollineation is called a double point [2].

Applications 3.4 : In many fields,equilibra or stability are fundmental concepts that can be discribed in terms of fixed points.

(a) for example,in ecomonosics,a nash eqilibrimum of game is a fixed point of the games best response correspondance.However,in physics,more precisely in the theory of phase Transitions,linearisation near an unstable fixed point has led to welsons Nobel prize –wining work inventing the renormalization group,and to the mathematical explination of the term "critical phenomenoon".

(b) In compliers,fixed point computions are used for whole program analysics,which are often reqired to do code opotimmization the vector of pageRank valuesof all web pages is the fixed point of a linear transformation derived from the world wide webs link structre.

(c) the concept of fixed point can be used to define the convergence of a function.

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