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Differential quadrature method for singularly perturbed differential-difference equations with small delay in convection term

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Abstract: In this paper, Differential Quadrature Method (DQM) is presented for finding the numerical solution of singularly perturbed differential-difference equations with small delay in convection term. Such problems are associated with the study of bistable devices, variational problems in control theory and the first exit time problems in the modelling of the determination of expected time for the generation of action potential in nerve cells by random synaptic inputs in dendrites. The Differential Quadrature Method is an efficient descritization technique in solving initial and /or boundary value problems accurately using a considerably small number of non-uniform grid points. We have used the Lagrange's interpolation technique to interpolate the solution values at uniform points. The derived Lagrange's interpolation polynomial is capable of producing almost the same accuracy as obtained in the DQM solution at non-uniform grid points. To demonstrate the applicability of these methods, we have solved the model example problems and compared the computational results with the exact solutions. Comparisons showed that the method is capable of producing highly accurate results with high efficiency.

Keywords: *Differential-difference equations; Differential Quadrature Method; Singular perturbation; Boundary value problem; Boundary layer*.

1. Introduction

A singularly perturbed differential- difference equation is an ordinary differential equation in which the highest derivative is multiplied by a small parameter and involving at least one delay or advance term. In recent papers the term negative or left shift and positive or right shift have been used for delay and advance respectively. Such problems arise frequently in the mathematical modelling of various practical phenomena, for example, in the modelling of several physical and biological phenomena like the optically bistable devices [20], description of the human pupil-light reflex [2], a variety of models for physiological processes or diseases and variational problems in control theory where they provide the best and in many cases the only realistic simulation of the observed phenomena [15].Any system involving a feedback control will almost always involve time delays. These arise because a finite time is required to sense information and then react to it. For a detailed discussion on differential-difference equation one may refer to the books and high level monographs: Bellen [1], Driver [22], Bellman and Cooke [25], Kuang [26]. It is well known fact that the solution of singularly perturbed differential-difference equation exhibits a multi scale character, that is, there are thin transition layer(s) where the solution varies rapidly, while away from the layers (s) the solution behaves regularly and varies slowly. Therefore, the numerical treatment of singularly perturbed differential-difference equations presents some major computational difficulties. If we apply the existing classical numerical methods for solving these problems, large oscillations may arise and pollute the solution in the entire interval because of the boundary layer behaviour. The smoothness of the solutions of such singularly perturbed differential-difference equations deteriorates when the parameter tends to zero. Lange and Miura[4-6] gave asymptotic approaches in the study of class of boundary-value problems for linear second-order differential-difference equations in which the highest order derivative is multiplied by small parameter. In [18], M.K. Kadalbajoo and K.K. Sharma described a numerical method to solve boundary value problems for singularly perturbed differential-difference equations with negative shift. They approximated the term containing delay by using Taylor series and applied the finite difference scheme. In [14], K.C. Patidar and K.K. Sharma approximated the term containing delay by Taylor series expansion and applied the non-standard finite difference methods for second order, linear, singularly perturbed differential-difference equations with small delay. In [16], the authors M.K. Kadalbajoo and D. Kumar presented a computational method based on piecewise uniform mesh for boundary value problems for nonlinear singularly perturbed differential-difference equations with small delay. In [17], they presented a fitted mesh B-spline collocation method for singularly perturbed differential-difference equations with small delay.In [19], the authors M.K. Kadalbajoo and V. P. Ramesh presented a hybrid method based on Shishkin mesh for the numerical solution of singularly perturbed delay differential equations. In [11], the authors J. Mahapatra and S. Natesan presented a numerical method for solving singularly perturbed differential-difference equations using grid equidistribution.

In this paper we have presented the differential quadrature method for finding the numerical solution of singularly perturbed differential-difference equations of second order where there is a small delay in the convection term. We have employed the Lagrange's interpolation technique to interpolate the solution values at uniform points. The DQM approximates the derivative with respect to a coordinate direction at a grid point by a weighted linear sum of all the functional values in that direction. The key to DQM is the determination of weighting coefficients for any order derivative discretization. To the best of the authors knowledge, the Differential Quadrature Method, where approximation of the derivatives have been based on a polynomial of high degree, has not been implemented for the singularly perturbed differential-difference equations of second order where there is a small delay in the convection term. This paper is organized as follows: Section 2 presents the description of the Differential Quadrature Method, including the formula for finding the weighting coefficients for any order derivative discretization and selection of sampling points. Section 3 presents the basic key procedure to solve differential equation with boundary conditions. Section 4 is devoted to the singularly perturbed differential-difference equations with small delay in convection term and its solution procedure by DQM in detail. In the section 5, we have considered two standard example problems having boundary layer at left/right end points of the interval and presented the computational results, show the accuracy and efficiency of the method. Based on the numerical experiments performed, some observations that include the alternative way of finding solution at uniform points with good accuracy have been presented in the section 6 under the heading discussion and conclusion.

2. Description of the method

The Differential Quadrature Method (DQM) is a numerical discretization technique for approximation of derivatives. It was initiated from the idea of integral quadrature. It was firstly introduced by Bellman et al.[23, 24] in the early of 1970s, and, since then, the technique has been extensively employed for finding the rapid and accurate solution of various problems in the field of applied and physical sciences :[3, 7-10, 21]. The basic idea of differential quadrature method is that the derivative of a function with respect to a space variable at a given point is approximated as a weighted linear sum of the functional values at all discrete points in the domain of that variable. In order to show the mathematical representation of the method, we consider a one dimensional field variable $f(x)$ prescribed in a field domain $a = x_1 \le x \le x_N = b$. Let $f_i = f(x_i)$ be the function values specified in a finite set of N discrete points $x_i(i = 1,2,......,N)$ of the field domain. Next, consider the value of the function derivative $d^m f / dx^m$ at some discrete points x_i, and let it be expressed as a linearly weighted some of the function values.

$$f^{(m)}(x_i) = \frac{d^m f(x_i)}{dx^m} = \sum_{J=1}^{N} A_{ij}^{(m)} f_j \left(i = 1,2,.......,N\right) \tag{1}$$

where $A_{ij}^{(m)}$ are the weighting coefficients of the m^{th}-order derivative of the function associated with points x_i. Equation (1) the quadrature rule for a derivative is the essential basis of the Differential Quadrature Method. Thus using equation (1) for various order derivatives, one may write a given differential equation at each point of its solution domain and obtain the quadrature analog of the differential equation as a set of algebraic equations in terms of the N function values. These equations may be solved, in conjunction with the quadrature analog of the boundary conditions, to obtain the unknown function values provided that the weighting coefficients are known a priori. The weighting coefficients may be determined by some appropriate functional approximations; and the approximate functions are referred to as test functions. The primary requirements for the choices of the test functions are of differentiability and smoothness. That is, the test function of the differential equation must be differentiable at least up to the n^{th} derivative (here n is the highest order of the differential equation) and sufficiently smooth to be satisfied the condition of the differentiability. The accuracy of differential quadrature solution depends on the accuracy of the weighting coefficients and the choice of sampling points. Bellman et al. [23] suggested two methods to determine the weighting coefficients of the first order derivative. The first method solves an algebraic equation system and the second method uses a simple algebraic formulation, but with the coordinates of grid points chosen as the roots of the shifted Legendre polynomial. Unfortunately, when the order of the algebraic equation system is large, its matrix is ill-conditioned. Thus it is very difficult to compute the weighting coefficients for a large number of grid points. To overcome the difficulties of Bellman's methods in computing the weighting coefficients, many attempts have been made by researchers. One of the most useful methods is the one introduced by Quan and Chang [12, 13]. After that Shu's [Shu [7]] general approach which is based on the high order polynomial approximation and linear vector space analyses, was made available in the literature. This generalized approach computes the weighting coefficients of the first order derivative by a simple algebraic formulation without any restriction on choice of grid points, and the weighting coefficients of second and higher order derivatives by a recurrence relationship. In the DQM, It is supposed that the solution of a one–dimensional differential equation is approximated by a $N-$terms high degree polynomial:

$$f(x)=\sum_{k=1}^{N} c_k . x^{k-1} \qquad (2)$$

where c_k is a constant.

The generalized approach uses two sets of base polynomials to determine the weighting coefficients (Shu [7]). The first set of base polynomials is chosen as the Lagrange interpolated polynomials, which are written as

$$r_k(x)=\frac{M(x)}{(x-x_k)\,.M^{(1)}(x_k)}; \qquad (3)$$
$$k=1,2,......,N$$

where

$$M(x)=(x-x_1).(x-x_2).......(x-x_N)$$

and

$$M^{(1)}(x_k)=\prod_{j=1,\, j\neq k}^{N}(x_k-x_j)$$

being the first derivative of $M(x)$ at x_k. Here x_1, x_2 ,......, x_N are the coordinates of the grid points, can be chosen arbitrarily but distinct.

The polynomials

$$r_k(x)=x^{k-1},\ k=1,2,........,N \qquad (4)$$

are taken as the second set of base polynomials.

For simplicity, by setting

$$M(x) = N(x, x_k).(x - x_k),\ k = 1,2,....,N$$

with $N(x_i, x_j) = M^{(1)}(x_i).\delta_{ij}$, where δ_{ij} is the Kronecker operator, the equation (3) is simplified as:

$$r_k(x) = \frac{N(x,\ x_k)}{M^{(1)}(x_k)};\ k = 1, 2, ..., N \tag{5}$$

Substituting Eq. (5) into the Eq. (1) for $m = 1$ and using Eq. (4), Shu [7] obtained the following weighting coefficients of the first order derivative discretization.

$$A_{ij}^{(1)} = \frac{M^{(1)}(x_i)}{(x_i - x_j)M^{(1)}(x_j)},\ (i, j = 1,2,.......N;\quad i \neq j) \qquad A_{ii}^{(1)} = -\sum_{j=1;i\neq j}^{N} A_{ij}^{(1)}, \quad (i = 1,2,......N) \tag{6}$$

The Shu's (Shu [7]) recurrence formulation for determination of weighting coefficients for higher order derivatives discretization are given as

$$A_{ij}^{(m)} = m[A_{ii}^{(m-1)} A_{ij}^{(1)} - \frac{A_{ij}^{(m-1)}}{(x_i - x_j)}], \qquad A_{ii}^{(m)} = -\sum_{j=1;i\neq j}^{N} A_{ij}^{(m)} \ ,(i = 1,2,......N;\ m \geq 2) \tag{7}$$
$$(i, j = 1,2,......N;\ i \neq j; 2 \leq m \leq N - 1)$$

Obviously, Eqs. (6) and (7) offer an easy way of computing the weighting coefficients for any order derivative discretization. These explicit formulae's merit is that highly accurate weighting coefficients may be determined for any number of arbitrarily spaced sampling points.

2.1. Choice of sampling points

A convenient and natural choice for the sampling points is that of the equally spaced points. But the Differential Quadrature solutions usually deliver more accurate results with unequally spaced sampling points. A rational basis for the sampling points is provided by the zeros of the orthogonal polynomials. A well accepted kind of sampling points in the DQM is the so called Gauss-Lobatto-Chebyshev sampling points. For a domain specified by $a \leq x \leq b$ and discretised by a set of unequally spaced points (non-uniform grid), then the coordinate of any point i can be evaluated by:

$$x_i = a + \frac{1}{2}\left(1 - \cos\left(\frac{i-1}{N-1}\pi\right)\right)(b - a) \tag{8}$$

3. Application to differential equation

The basic key procedure in the DQM is to approximate the derivatives in a differential equation by equation (1). Substituting the equation (1) into the governing equations and equating both sides of the governing equations, we obtain simultaneous equations which can be solved by use of Gauss elimination or other methods. That is, DQM is composed of the following procedure:

(a) The function to be determined is replaced by a group of function values at a group of selected sampling points. Gauss-Lobatto-Chebyshev sampling points(8) are strongly recommended for numerical stability,

(b) Approximate derivatives in a differential equation by these N unknown function values.

(c) Form a system of linear equations and

(d) Solving the system of linear equation yields the desired unknowns.

The proper implementation of boundary condition is very important for the accurate numerical solution of differential equation. Essential and natural boundary condition can be approximated by DQM. Using the technique in solving differential equation, the governing equations are actually satisfied at each sampling point of the domain, so one has one equation for each point, for each unknown. In the resulting system of algebraic equation from the DQM, each boundary condition replaces the corresponding field equation. This procedure is straightforward when there is one boundary condition at each boundary and when we have distributed the sampling points so that there is one point at each boundary.

4. Application to singularly perturbed differential-difference equations with small delay in convection term

To show the applicability of DQM, we consider the boundary-value problems for a class of singularly perturbed differential-difference equations with small delay in convection term of the form:

$$\varepsilon y''(x) + a(x)y'(x-\delta) + b(x)y(x) = f(x); \quad 0 \le x \le 1 \qquad (9)$$

with

$$y(x) = \psi(x) \text{ on } -\delta \le x \le 0, \text{ and } y(1) = \lambda \qquad (10)$$

where ε is a singular perturbation parameter, $(0 < \varepsilon << 1)$ and δ is also a small shifting parameter, $0 < \delta = o(\varepsilon)$ such that $\varepsilon - \delta\, a(x) > 0$ for all $x \in [0,1]$ .The functions $a(x), b(x)$,

$f(x)$, and $\psi(x)$ are assumed to be sufficiently continuously differentiable functions in $[0,1]$ and λ is a given constant.

The boundary-value problem (9) with(10) exhibits the layer behaviour

(i) on the left side of the interval $[0,1]$, when $a(x) \geq \beta > 0$ throughout the interval $[0,1]$, where β is some positive constant

(ii) on the right side of the interval $[0,1]$, when $a(x) \leq \alpha < 0$ throughout the interval $[0,1]$, where α is some negative constants.

(iii) in the turning point region when $a(x)$ changes sign in the interval $[0,1]$.

In this paper, we will consider the first two cases in which the boundary value problem (9) with (10) exhibits the layer behaviour at left/right end points of the interval considered. Since the solution of the boundary-value problem (9) with (10) is continuous and continuously differentiable, so expanding the term containing delay δ i.e. the term $y'(x-\delta)$ by Taylor series, we obtain

$$y'(x-\delta) \approx y'(x) - \delta\, y''(x), \qquad (11)$$

Using Eq.(11) in Eq.(9)with (10) we obtain

$$(\varepsilon - \delta\, a(x))\, y''(x) + a(x) y'(x) + b(x) y(x) \approx f(x); \quad 0 \leq x \leq 1 \qquad (12)$$

with $y(0) \approx \psi(0)$ and $y(1) \approx \lambda$ (13)

Since the Eq. (12) with (13) is an approximate version of Eq.(9) with (10), it is reasonable to use different notation (say $w(x)$) for the solution of this approximate differential equation. Thus the problem (12) with (13) results into the following more general singularly perturbed boundary value problem:

$$(\varepsilon - \delta\, a(x))\, w''(x) + a(x) w'(x) + b(x) w(x) = f(x); \quad 0 \leq x \leq 1 \qquad (14)$$

with $w(0) = \psi(0)$ and $w(1) = \lambda$ (15)

Since we have considered the problem in which shift δ is of small order of ε, the derivatives of the solutions of such problems have the bonds $\left|y^{(k)}(x)\right| \leq M/(o(\varepsilon^k)), k \in I^+$, for a positive constant M, which implies that $\delta\left|y'(x)\right| \leq M$, etc. This validates the use of Taylor expansions in the original problem.

We will solve this boundary value problem (14) with (15) by DQM to approximate the solution of boundary-value problem (9) with (10) over the interval $[0,1]$. For finding the solution of the Eq. (14) with the boundary conditions (15) by DQM, we have followed the following procedure/steps:

(i) Discretize the interval $[0,1]$, such that $0 = x_1 < x_2 < x_3 < .... < x_N = 1$ where, N is the number of Gauss-Lobatto-Chebyshev sampling points obtained from Eq. (8). Denote $w_i = w(x_i), a_i = a(x_i)$ and $f_i = f(x_i)$ etc.

(ii) Apply the DQM to approximate the derivatives in the Eq. (14), that leads to the following discretized form of the equation:

$$(\varepsilon - \delta a_i)\sum_{j=1}^{N} A_{i,j}^{(2)} w_j + a_i \sum_{j=1}^{N} A_{i,j}^{(1)} w_j + b_i w_i - f_i = 0.\ i = 1,2,.........., N \quad (16)$$

with $w_1 = \psi(0)$ and $w_N = \lambda$ (17)

(iii) Apply the Eq. (16) at all interior points: $x_i, (i = 2,3,..........., N-1)$, that leads to a system of $(N-2)$ equations with N unknowns.

(iv) Use the boundary values for w_1 and w_N from Eq. (17) in the obtained system of equations from step (iii) to get another system of $(N-2)$ equations with $(N-2)$ unknowns $(w_i, i = 2,3,........., N-1)$.

(v) Solve the system of equations obtained in step (iv) for the unknowns $(w_i, i = 2,3,........., N-1)$

(vi) Use the given boundary values to get the complete solution.

We have applied the Gaussian elimination method with partial pivoting and employed the double precision Fortran, to solve the obtained system of linear equations in the step (iv), for the unknowns $(w_i, i = 2,3,........., N-1)$. Hereafter the solution so obtained will be referred to as **DQM** solution (Differential Quadrature Method solution).

5. Numerical Experiments

To demonstrate the applicability and efficiency of the **DQM** with Gaussian elimination and Lagrange's interpolation techniques, we have applied it to two standard example problems having a boundary layer at left /right end points of the interval. The first example problem is the problem having boundary layer at left end point of the interval while second example problem is the problem having boundary layer at right end point of the interval. We have solved the first example problem for different values of δ for fixed $\varepsilon = 0.01$ and $\varepsilon = 0.001$ respectively and the second example problem for different values of ε with $\delta = 0.5\varepsilon$. Note that for the considered example problems, the DQM results in the tables are given in terms of

Maximum Absolute Error (M.A.E.) at uniform grids $x_i = ih, (i = 0,1,2,......,K)$, with $h = 1/K$ for the considered interval [0,1], which have been interpolated through the use of Lagrange's interpolation polynomial. For the derivation of this polynomial, we have used the DQM results $(x_i, y_i), i = 1,2,......,N$, where $y_i, (i = 1,2,......,N)$ are the value of y at non-uniform grid points (Gauss-Lobatto-Chebyshev points) $x_i, (i = 1,2,........,N)$ obtained from (8). To show the accuracy and efficiency of the method with non-uniform grid points (Gauss-Lobatto-Chebyshev points) $x_i, (i = 1,2,........,N)$ obtained from (8), we have also given the computational results in terms of Maximum Absolute Error in the tables for the examples for different values of N, δ and small parameter: ε.

Example 1: Consider the following singularly perturbed differential-difference equations from [14, 18]:

$$\varepsilon y''(x) + y'(x-\delta) - y(x) = 0$$

with $y(x) = 1, \quad -\delta \le x \le 0, \quad y(1) = 1.$

For this example we have $a(x) = 1, b(x) = -1$ and $f(x) = 0.$ Further, we have a boundary layer at $x = 0$, i.e., on the left side of the interval $[0,1]$. The exact solution is given by:

$$y(x) = [(1 - \exp(m_2))\exp(m_1 x)$$
$$-(1 - \exp(m_1))\exp(m_2 x)]$$
$$/[\exp(m_1) - \exp(m_2)]$$

where;

$$m_1 = \frac{-1 + \sqrt{1 + 4(\varepsilon - \delta)}}{2(\varepsilon - \delta)} \quad \text{and} \quad m_2 = \frac{-1 - \sqrt{1 + 4(\varepsilon - \delta)}}{2(\varepsilon - \delta)}$$

The computational results are presented in Table 1(a), 1(b),1(c),1(d)for different values of N, K, δ and small parameter: ε.

Example 2: Consider the following singularly perturbed differential-difference equations from [14, 18]:

$$\varepsilon y''(x) - y'(x-\delta) - y(x) = 0$$

with $y(x) = 1, -\delta \le x \le 0, \; y(1) = -1.$

For this example we have $a(x) = -1,\ b(x) = -1$ and $f(x) = 0.$ Further we have a boundary layer at $x = 1,$ i.e., on the right side of the interval $[0,1].$

The exact solution is given by:

$$y(x) = [-(1+\exp(m_2))\exp(m_1 x)$$
$$+(1+\exp(m_1))\exp(m_2 x)]$$
$$/[\exp(m_1) - \exp(m_2)]$$

where;

$$m_1 = \frac{1+\sqrt{1+4(\varepsilon+\delta)}}{2(\varepsilon+\delta)} \text{ and } \quad m_2 = \frac{1-\sqrt{1+4(\varepsilon+\delta)}}{2(\varepsilon+\delta)}$$

The computational results are presented in Table 2(a) and 2(b), for different values of $N,\ K,\ \delta$ and ε.

6. Discussions and Conclusions

In this paper, we have presented the Differential Quadrature Method (DQM) for finding the numerical solution of linear, second order singularly perturbed differential-difference equations with small delay in convection term, provided the shift δ is of small order of singular perturbation parameter ε.We have used the Gaussian elimination and Lagrange's interpolation techniques for finding the solution at non-uniform and uniform points respectively. We have solved the two model example problems having a boundary layer at left/right end points of the interval and presented the computational results in tables. We have given here only a few values although the solutions can be computed at desired number of uniform points with almost the same higher accuracy as obtained in the DQM solution.

Based on the numerical experiments performed in this paper the following observations are made:

1. DQM has the capability of solving the considered problems and producing highly accurate results with small number of sampling points and minimal computational effort.

2. As N increases, there is slight increase and decrease in the accuracy but with reasonably good accuracy. This observation can easily be observed from the

computational results presented in the tables 1(a) to 1(d)for example problem-1 and 2(a), 2(b) for example problem-2.

3. When δ is very close to ε, The number of sampling points needed for better accuracy is more with respect to the smaller δ's. This observation can be seen in the tables 1(a) to 1(d) for the example problem-1. Further, the number of sampling points needed for achieving higher accuracy with smaller ε's is more with respect to the larger one. This observation can be seen in the tables 1(a), 1(c) and 2(a), 2(b) for the example problem-1 and 2 respectively

4. The derived Lagrange's interpolation polynomial is capable of producing almost the same accuracy as obtained in the DQM solution. That is, if the DQM solution is associated with some fixed decimal accuracy (say, for example, four decimal accuracy) for any fixed N,ε and δ then the derived Lagrange's interpolation polynomial with this solution is capable of producing almost the same(four decimal) accuracy for all the values of K. This observation can easily be observed from the computational results presented in the tables for example problem-1 and 2 respectively.

5. The single, derived Lagrange's interpolation polynomial can be used for finding the solution at uniform points for the tiny values of the singular perturbation parameter: ε and the delay δ, where δ is small order of ε. In addition to this, it can also be used for finding the solution at uniform points for all the different values of delay δ, where ε is fixed and δ is of small order of ε. As observed, better accuracy with good values of K is obtained when the Lagrange's interpolation polynomial is derived with the use of those data points for which DQM solution is obtained by choosing $\varepsilon \le 10^{-3}$. The values of K increases when ε decreases.

To show the accuracy obtained in this observation for tiny values of ε and their corresponding δ's, we have computed the solution for different values of K from derived Lagrange's interpolation polynomial using the DQM solution($(x_i, y_i), i = 1,2,......,N$), with $N = 126, \varepsilon = 10^{-3}, \delta = 5.0E-04$ and $N = 325, \varepsilon = 10^{-4}$, $\delta = 3.0E-05$ respectively, where $y_i, i = 1,2,......,N$ are the values of y at non-uniform grid points (Gauss-Lobatto-Chebyshev points) $x_i, i = 1,2,........,N$ obtained from (8)), and presented the computational results in terms of maximum absolute errors in the table- 1.1(c) and 1.1(d) for the example problem-1. It can easily be observed that the value of K having good accuracy is much more for $\varepsilon = 10^{-4}$ than for $\varepsilon = 10^{-3}$ and when K increases accuracy decreases. Further, the maximum absolute errors are almost the same for a fixed value of K for all tiny values of ε and their corresponding δ's.

Further, to show the accuracy obtained in the solution for different values of δ only, we have computed the solution for different values of K from derived Lagrange's interpolation polynomial using the DQM solution ((x_i, y_i), $i = 1,2,......,N$), with

$N = 168, \varepsilon = 10^{-3}, \delta = 0.0004$ and $N = 345, \varepsilon = 10^{-4}, \delta = 0.00004$ respectively, where $y_i, i = 1,2,......, N$ are the value of y at non-uniform grid points (Gauss-Lobatto-Chebyshev points) $x_i, i = 1,2,........, N$ obtained from (8)), and presented the computational results in terms of maximum absolute errors in the table-1.1(a) and 1.1(b),for the example problem-1. It can be easily observed that the value of K having good accuracy is much more for $\varepsilon = 10^{-4}$ than for $\varepsilon = 10^{-3}$ and when K increases accuracy decreases. Obviously, the above observations provide a simple alternative technique that uses single derived Lagrange's interpolation polynomial for finding the solution at a good number of uniform points with reasonably good accuracy for the tiny values of the singular perturbation parameter: ε and their corresponding δ's. In addition to this, the technique is also able to provide accurate results for all the different values of δ, where ε is fixed. Thus, the DQM with Lagrange's interpolation technique provides us two different efficient way of finding the solution at uniform points for the problems considered.

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Table 1(a). Maximum Absolute Error in the DQM solution for the example **1** at non-uniform grid points (Gauss-Lobatto-Chebyshev points) $x_i, i = 1,2,........, N$ obtained from (8), with $\varepsilon = 0.01$ and different values of N and δ.

$\varepsilon = 0.01$				
δ	$N = 35$	$N = 45$	$N = 65$	$N = 85$
0.001	.4337E-05	.5436E-06	.2315E-05	.7807E-06
0.002	.1307E-04	.4824E-06	.2092E-05	.7115E-06
0.003	.4217E-04	.5725E-06	.1849E-05	.6377E-06
0.004	.1416E-03	.1462E-05	.1634E-05	.5589E-06
0.005	.4919E-03	.8170E-05	.1429E-05	.4751E-06
0.006	.1782E-02	.5954E-04	.1196E-05	.3862E-06
0.007	.6807E-02	.4776E-03	.8579E-06	.2923E-06
0.008	.2751E-01	.4280E-02	.3659E-04	.2707E-06

Table 1(b) Maximum Absolute Error in the solution for the example **-1**(computed from derived Lagrange's interpolation polynomial at uniform points ($x_i = ih, (i = 0,1,2,......, K),$ where $h = 1/K$) with $\varepsilon = 0.01$ and different values of N, K and δ.

δ	$\varepsilon = 0.01$							
	$N = 35$		$N = 45$		$N = 65$		$N = 85$	
	$K =$ 100	$K =$ 100000	$K =$ 100	$K =$ 100000	$K =$ 100	$K =$ 100000	$K =$ 100	$K =$ 100000
0.001	.4996 E-05	.5042E-05	.5613 E-06	.5631E-06	.2277 E-05	.2385E-05	.8044 E-06	.8051E-06
0.002	.1480 E-04	.1486E-04	.4993 E-06	.4997E-06	.2078 E-05	.2217E-05	.7427 E-06	.7429E-06
0.003	.4658 E-04	.4668E-04	.5757 E-06	.5932E-06	.1814 E-05	.1982E-05	.6490 E-06	.6490E-06
0.004	.1525 E-03	.1529E-03	.1521 E-05	.1613E-05	.1561 E-05	.1766E-05	.5783 E-06	.5786E-06
0.005	.5182 E-03	.5207E-03	.8538 E-05	.8897E-05	.1263 E-05	.1497E-05	.4771 E-06	.4782E-06
0.006	.1847 E-02	.1855E-02	.6218 E-04	.6314E-04	.9875 E-06	.1247E-05	.4007 E-06	.4025E-06
0.007	.6914 E-02	.7006E-02	.4944 E-03	.4955E-03	.8691 E-06	.9948E-06	.2962 E-06	.3582E-06
0.008	.2772 E-01	.2802E-01	.4381 E-02	.4381E-02	.3774 E-04	.3779E-04	.2522 E-06	.3472E-06

Table 1(c). Maximum Absolute Error in the DQM solution for the example **-1** at non-uniform grid points (Gauss-Lobatto-Chebyshev points) $x_i, i = 1,2,........,N$ obtained from (8), with $\varepsilon = 0.001$ and for different values of N and δ.

δ	$\varepsilon = 0.001$					
	$N = 85$	$N = 118$	$N = 148$	$N = 178$	$N = 218$	$N = 248$
0.0001	.2653E-03	.1066E-05	.1354E-05	.1736E-05	.9968E-06	.1100E-05
0.0002	.5785E-03	.2496E-05	.1247E-05	.1669E-05	.9538E-06	.9971E-06
0.0003	.1279E-02	.8742E-05	.1123E-05	.1588E-05	.9129E-06	.8880E-06
0.0004	.2877E-02	.3591E-04	.9878E-06	.1488E-05	.8593E-06	.8023E-06
0.0005	.6594E-02	.1558E-03	.2476E-05	.1365E-05	.8097E-06	.7700E-06
0.0006	.1539E-01	.7047E-03	.2263E-04	.1188E-05	.7557E-06	.7162E-06
0.0007	.3629E-01	.3369E-02	.2319E-03	.1011E-04	.6667E-06	.6237E-06
0.0008	.8392E-01	.1729E-01	.2638E-02	.2950E-03	.9531E-05	.7179E-06

Table 1(d) Maximum Absolute Error in the solution for the example **1** (computed from derived Lagrange's interpolation polynomial at uniform points ($x_i = ih, (i = 0,1,2,......,K)$ where $h = 1/K$) with $\varepsilon = 0.001$ and different values of N, K and δ.

δ	$\varepsilon = 0.001$							
	$N = 85$		$N = 148$		$N = 178$		$N = 248$	
	$K =$ 100	$K =$ 100000	$K =$ 100	$K =$ 100000	$K =$ 100	$K =$ 100000	$K =$ 100	$K =$ 100000
0.0001	.2653 E-03	.2696E-03	.8653 E-06	.1457E-05	.7263 E-06	.1723E-05	.1032 E-05	.1093E-05
0.0002	.5782 E-03	.5871E-03	.1074 E-05	.1383E-05	.6399 E-06	.1643E-05	.9270 E-06	.9918E-06
0.0003	.1275 E-02	.1297E-02	.7503 E-06	.1251E-05	.5534 E-06	.1595E-05	.8026 E-06	.9136E-06
0.0004	.2866 E-02	.2912E-02	.8094 E-06	.1130E-05	.4767 E-06	.1474E-05	.7035 E-06	.8657E-06
0.0005	.6550 E-02	.6663E-02	.2486 E-05	.2530E-05	.3968 E-06	.1364E-05	.5867 E-06	.8427E-06
0.0006	.1522 E-01	.1553E-01	.2265 E-04	.2290E-04	.5935 E-06	.1306E-05	.4541 E-06	.8350E-06
0.0007	.3572 E-01	.3653E-01	.2297 E-03	.2337E-03	.1015 E-04	.1018E-04	.3532 E-06	.7328E-06
0.0008	.8270 E-01	.8424E-01	.2557 E-02	.2655E-02	.2909 E-03	.2971E-03	.6832 E-06	.7389E-06

Table 2(a). Maximum Absolute Error in the DQM solution at non-uniform grid points (Gauss-Lobatto-Chebyshev points) $x_i, i = 1,2,........,N$ obtained from (8) for different values of N and ε where $\delta = 0.5\varepsilon$, for the example **2**.

ε	$\delta = 0.5\varepsilon$				
	$N = 45$	$N = 55$	$N = 65$	$N = 75$	$N = 85$
2^{-1}	.1995E-05	.6655E-05	.2104E-04	.5996E-05	.8225E-05
2^{-2}	.2279E-05	.6809E-05	.2105E-04	.6150E-05	.8470E-05
2^{-3}	.2291E-05	.7307E-05	.1904E-04	.6583E-05	.8850E-05
2^{-4}	.2060E-05	.7292E-05	.1516E-04	.7118E-05	.8766E-05
2^{-5}	.1531E-05	.5744E-05	.1099E-04	.7176E-05	.7021E-05
2^{-6}	.1067E-05	.3466E-05	.7804E-05	.6222E-05	.4743E-05
2^{-7}	.9788E-06	.1697E-05	.5352E-05	.4422E-05	.2993E-05
2^{-8}	.3244E-05	.7234E-06	.3409E-05	.2775E-05	.1845E-05
2^{-9}	.1203E-02	.5507E-04	.3276E-05	.1826E-05	.1110E-05
2^{-10}	.3194E-01	.5874E-02	.8402E-03	.9373E-04	.8615E-05

Table 2(b) Maximum Absolute Error in the solution (computed from fitted Lagrange's interpolation polynomial) for uniform points: $x_i = ih, (i = 0,1,2,......,K)$ with $h = 1/K$ for different values of N and ε where $\delta = 0.5\varepsilon$, for the example **2**.

ε	$\delta = 0.5\varepsilon$							
	$N = 45$		$N = 55$		$N = 75$		$N = 85$	
	$K =$ 100	$K =$ 100000	$K =$ 100	$K =$ 100000	$K =$ 100	$K =$ 100000	$K =$ 100	$K =$ 100000
2^{-1}	.1954 E-05	.1956E-05	.6642 E-05	.6658E-05	.5976 E-05	.6002E-05	.8323 E-05	.8327E-05
2^{-2}	.2368 E-05	.2374E-05	.6914 E-05	.6925E-05	.6255 E-05	.6265E-05	.8445 E-05	.8459E-05
2^{-3}	.2261 E-05	.2261E-05	.7235 E-05	.7255E-05	.6470 E-05	.6491E-05	.8926 E-05	.8986E-05
2^{-4}	.2187 E-05	.2191E-05	.7420 E-05	.7420E-05	.7246 E-05	.7287E-05	.8606 E-05	.8620E-05
2^{-5}	.1930 E-05	.1932E-05	.6202 E-05	.6235E-05	.7681 E-05	.7682E-05	.6585 E-05	.6594E-05
2^{-6}	.1709 E-05	.1996E-05	.4140 E-05	.4141E-05	.6807 E-05	.6849E-05	.4135 E-05	.4144E-05
2^{-7}	.3621 E-05	.4266E-05	.4152 E-05	.4383E-05	.6921 E-05	.6992E-05	.2705 E-05	.5748E-05
2^{-8}	.4282 E-05	.5717E-05	.2188 E-05	.4026E-05	.1554 E-05	.9058E-05	.3165 E-05	.8559E-05
2^{-9}	.1250 E-02	.1252E-02	.6238 E-04	.6270E-04	.6869 E-05	.1454E-04	.7575 E-05	.1464E-04
2^{-10}	.3236 E-01	.3248E-01	.5965 E-02	.5978E-02	.9679 E-04	.9841E-04	.2176 E-04	.3762E-04

Table 1.1(a). Maximum absolute error in the solution K (computed from derived Lagrange's interpolation polynomial using the DQM solution ($(x_i, y_i), i = 1,2,......,N$), with $N = 168, \varepsilon = 10^{-3}, \delta = 0.0004,$ where $y_i, i = 1,2,......,N$ are the value of y at non-uniform grid points (Gauss-Lobatto-Chebyshev points) $x_i, i = 1,2,........,N$ obtained from (8)), for different values of δ, for the example **1.**

δ	$K = 100$	$K = 200$	$K = 300$	$K = 900$
0.0001	.1197E-03	.2390E-02	.1319E-01	.8467E-01
0.0002	.7600E-04	.1137E-02	.7402E-02	.5838E-01
0.0003	.3737E-04	.3834E-03	.2987E-02	.3003E-01
0.0005	.3662E-04	.1585E-03	.1669E-02	.3069E-01
0.0006	.7333E-04	.2215E-03	.2356E-02	.5989E-01
0.0007	.1101E-03	.2605E-03	.2535E-02	.8362E-01
0.0008	.1468E-03	.2973E-03	.2581E-02	.9677E-01
0.0009	.1836E-03	.3341E-03	.2618E-02	.9924E-01

Table 1.1(b). Maximum absolute error in the solution for different values of K (computed from derived Lagrange's interpolation polynomial using the DQM solution ((x_i, y_i), $i = 1,2,......,N$), with $N = 345, \varepsilon = 10^{-4}, \delta = 0.00004,$ where $y_i, i = 1,2,......,N$ are the value of y at non-uniform grid points (Gauss-Lobatto-Chebyshev points) $x_i, i = 1,2,........,N$ obtained from (8)), for different values of δ, for the example **1.**

δ	$K = 1500$	$K = 2500$	$K = 4500$	$K = 6500$
0.00001	.3981E-03	.6556E-02	.3800E-01	.6571E-01
0.00002	.1630E-03	.3389E-02	.2379E-01	.4370E-01
0.00003	.1181E-03	.1213E-02	.1092E-01	.2151E-01
0.00005	.1197E-03	.6669E-03	.8090E-02	.1954E-01
0.00006	.1222E-03	.8539E-03	.1307E-01	.3518E-01
0.00007	.1255E-03	.8852E-03	.1514E-01	.4494E-01
0.00008	.1290E-03	.8899E-03	.1551E-01	.4840E-01
0.00009	.1326E-03	.8936E-03	.1553E-01	.4869E-01

Table 1.1(c). Maximum absolute error in the solution for different values of K (computed from derived Lagrange's interpolation polynomial using the DQM solution ((x_i, y_i), $i = 1,2,......,N$), with $N = 126, \varepsilon = 10^{-3}, \delta = 5.0E-04$, where $y_i, i = 1,2,......,N$ are the value of y at non-uniform grid points (Gauss-Lobatto-Chebyshev points) $x_i, i = 1,2,........,N$ obtained from (8)), for different values of ε and δ, for the example **1**.

ε, δ	$K = 100$	$K = 300$	$K = 400$	$K = 900$	$K = 1000$
$\varepsilon = 10^{-4}$, δ =1.0E-05	.1979E-03	.9678E-03	.4415E-02	.6856E-01	.8560E-01
$\varepsilon = 10^{-4}$, δ =3.0E-05	.2053E-03	.9752E-03	.4422E-02	.6857E-01	.8561E-01
$\varepsilon = 10^{-4}$, δ =6.0E-05	.2163E-03	.9862E-03	.4433E-02	.6858E-01	.8562E-01
$\varepsilon = 10^{-6}$, δ =2.0E-07	.2307E-03	.1001E-02	.4448E-02	.6860E-01	.8564E-01
$\varepsilon = 10^{-6}$, δ =5.0E-07	.2308E-03	.1001E-02	.4448E-02	.6860E-01	.8564E-01
$\varepsilon = 10^{-6}$, δ =7.0E-07	.2309E-03	.1001E-02	.4448E-02	.6860E-01	.8564E-01
$\varepsilon = 10^{-9}$, δ =3.0E-10	.2310E-03	.1001E-02	.4448E-02	.6860E-01	.8564E-01
$\varepsilon = 10^{-9}$, δ =6.0E-10	.2310E-03	.1001E-02	.4448E-02	.6860E-01	.8564E-01
$\varepsilon = 10^{-9}$, δ =8.0E-10	.2309E-03	.1001E-02	.4448E-02	.6860E-01	.8564E-01
$\varepsilon = 10^{-10}$, δ =2.0E-11	.2311E-03	.1001E-02	.4448E-02	.6860E-01	.8564E-01
$\varepsilon = 10^{-10}$, δ =5.0E-11	.2311E-03	.1001E-02	.4448E-02	.6860E-01	.8564E-01
$\varepsilon = 10^{-10}$, δ =8.0E-11	.2310E-03	.1001E-02	.4448E-02	.6860E-01	.8564E-01
$\varepsilon = 10^{-12}$, δ =3.0E-13	.2287E-03	.9986E-03	.4446E-02	.6860E-01	.8564E-01
$\varepsilon = 10^{-12}$, δ =6.0E-13	.2433E-03	.1013E-02	.4460E-02	.6861E-01	.8565E-01
$\varepsilon = 10^{-12}$, δ =8.0E-13	.1414E-03	.9111E-03	.4358E-02	.6851E-01	.8555E-01
$\varepsilon = 10^{-14}$, δ =1.0E-15	.1568E-03	.7068E-03	.4154E-02	.6831E-01	.8534E-01
$\varepsilon = 10^{-14}$, δ =4.0E-15	.1568E-03	.7068E-03	.4154E-02	.6831E-01	.8534E-01
$\varepsilon = 10^{-14}$, δ =7.0E-15	.1568E-03	.7067E-03	.4154E-02	.6831E-01	.8534E-01
$\varepsilon = 10^{-14}$, δ =9.0E-15	.1569E-03	.7066E-03	.4154E-02	.6831E-01	.8534E-01

Table 1.1(d). Maximum absolute error in the solution for different values of K (computed from derived Lagrange's interpolation polynomial using the DQM solution ((x_i, y_i), $i=1,2,......,N$), with $N=325, \varepsilon=10^{-4}$, $\delta=3.0E-05$ where $y_i, i=1,2,......,N$ are the value of y at non-uniform grid points (Gauss-Lobatto-Chebyshev points) $x_i, i=1,2,........,N$ obtained from (8)), for different values of ε and δ, for the example **1**.

ε, δ	$K=1000$	$K=2000$	$K=3000$	$K=5000$	$K=7000$
$\varepsilon=10^{-7}$, δ =2.0E-08	.1130E-03	.5520E-03	.5429E-02	.3631E-01	.8217E-01
$\varepsilon=10^{-7}$, δ =5.0E-08	.1130E-03	.5520E-03	.5429E-02	.3631E-01	.8217E-01
$\varepsilon=10^{-7}$, δ =9.0E-08	.1130E-03	.5520E-03.	.5429E-02	.3631E-01	.8217E-01
$\varepsilon=10^{-10}$, δ =3.0E-11	.1130E-03	.5520E-03	.5429E-02	.3631E-01	.8217E-01
$\varepsilon=10^{-10}$, δ =6.0E-11	.1130E-03	.5520E-03	.5429E-02	.3631E-01	.8217E-01
$\varepsilon=10^{-10}$, δ =8.0E-11	.1130E-03	.5520E-03	.5429E-02	.3631E-01	.8217E-01
$\varepsilon=10^{-11}$, δ =2.0E-12	.1100E-03	.5490E-03	.5426E-02	.3631E-01	.8217E-01
$\varepsilon=10^{-11}$, δ =5.0E-12	.1130E-03	.5520E-03	.5429E-02	.3631E-01	.8217E-01
$\varepsilon=10^{-11}$, δ =8.0E-12	.1050E-03	.5439E-03	.5420E-02	.3630E-01	.8216E-01
$\varepsilon=10^{-13}$, δ =3.0E-14	.3497E-03	.3497E-03	.5134E-02	.3602E-01	.8188E-01
$\varepsilon=10^{-13}$, δ =6.0E-14	.3497E-03	.3497E-03	.5134E-02	.3602E-01	.8188E-01
$\varepsilon=10^{-13}$, δ =7.0E-14	.3497E-03	.3497E-03	.5134E-02	.3602E-01	.8188E-01
$\varepsilon=10^{-14}$, δ =1.0E-15	.3497E-03	.3497E-03	.5134E-02	.3602E-01	.8188E-01
$\varepsilon=10^{-14}$, δ =4.0E-15	.3497E-03	.3497E-03	.5134E-02	.3602E-01	.8188E-01
$\varepsilon=10^{-14}$, δ =7.0E-15	.3497E-03	.3497E-03	.5134E-02	.3602E-01	.8188E-01
$\varepsilon=10^{-14}$, δ =9.0E-15	.3498E-03	.3498E-03	.5134E-02	.3602E-01	.8188E-01